Exercises from Section 1.2.3

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1. [*10*] The text says that $a_1 + a_2 + \cdots + a_0 = 0$. What then, is $a_2 + \cdots + a_0$?

The identities

$$\sum_{1 \le k \le 0} a_k = 0 = a_1 + \sum_{2 \le k \le 0} a_k$$

imply that

$$\sum_{2 \le k \le 0} a_k = a_2 + \cdots + a_0 = -a_1.$$

2. [*01*] What does the notation $\sum_{1 \le j \le n} a_j$ mean, if $n = 3.14$?

The notation $\sum_{1 \le j \le n} a_j$ if $n = 3.14$ represents the sum of all a_j such that $1 \le j \le n = 3.14$, or equivalently, $j \in \{1, 2, 3\}$. That is,

$$\sum_{1 \le j \le n} a_j = a_1 + a_2 + a_3.$$

▶ **3.** [*13*] Without using the $\sum$-notation, write out the equivalent of

$$\sum_{0 \le n \le 5} \frac{1}{2n+1}$$

and also the equivalent of

$$\sum_{0 \le n^2 \le 5} \frac{1}{2n^2+1}.$$

Explain why the two results are different, in spite of rule (b).

For the first notation,

$$\sum_{0 \le n \le 5} \frac{1}{2n+1} = \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11},$$

while for the second notation,

$$\sum_{0 \le n^2 \le 5} \frac{1}{2n^2+1} = \frac{1}{9} + \frac{1}{3} + \frac{1}{1} + \frac{1}{3} + \frac{1}{9}.$$

The two results are different, in spite of rule (b), as $P(n) = n^2$ is not a permutation, since P is neither one-to-one ($-1^2 = 1^2$ serves as a counterexample) nor onto (there is no integer n such that $n^2 = 3$).

4. [*10*] Without using the $\sum$-notation, write out the equivalent of each side of Eq. (10) as a sum of sums for the case $n = 3$.

For $n = 3$, the equivalent of the left side of Eq. (10) is

$$\sum_{i=1}^{3} \sum_{j=1}^{i} a_{ij} = (a_{11}) + (a_{21} + a_{22}) + (a_{31} + a_{32} + a_{33})$$

and of the right side is

$$\sum_{j=1}^{3}\sum_{i=j}^{3} = (a_{11} + a_{21} + a_{31}) + (a_{22} + a_{32}) + (a_{33}).$$

▶ **5.** [*HM20*] Prove that rule (a) is valid for arbitrary infinite series, provided the series converge.

Proposition. *The distributive law, for the products of sums* $\left(\sum_{R(i)} a_i\right)\left(\sum_{S(j)} b_j\right) = \sum_{R(i)}\left(\sum_{S(j)} a_i b_j\right)$ *holds for arbitrary infinite series, provided the sums converge.*

Proof. Assume that we have two infinite series a_i, b_j whose sums converge; that is, that

$$\sum_{R(i)} a_i = \left(\lim_{n\to\infty} \sum_{\substack{R(i)\\ 0\le i<n}} a_i\right) + \left(\lim_{n\to\infty} \sum_{\substack{R(i)\\ -n\le i<0}} a_i\right) = \alpha$$

and

$$\sum_{S(j)} b_j = \left(\lim_{n\to\infty} \sum_{\substack{S(j)\\ 0\le j<n}} b_j\right) + \left(\lim_{n\to\infty} \sum_{\substack{S(j)\\ -n\le j<0}} b_j\right) = \beta$$

for arbitary limits α, β. We must show that

$$\left(\sum_{R(i)} a_i\right)\left(\sum_{S(j)} b_j\right) = \sum_{R(i)}\left(\sum_{S(j)} a_i b_j\right).$$

But

$$\begin{aligned}
\left(\sum_{R(i)} a_i\right)\left(\sum_{S(j)} b_j\right) &= \lim_{n\to\infty}\left(\left(\sum_{\substack{R(i)\\ 0\le i<n}} a_i + \sum_{\substack{R(i)\\ -n\le i<0}} a_i\right)\left(\sum_{\substack{S(j)\\ 0\le j<n}} b_j + \sum_{\substack{S(j)\\ -n\le j<0}} b_j\right)\right)\\
&= \lim_{n\to\infty}\left(\left(\sum_{\substack{R(i)\\ 0\le i<n}} a_i\right)\left(\sum_{\substack{S(j)\\ 0\le j<n}} b_j + \sum_{\substack{S(j)\\ -n\le j<0}} b_j\right) + \left(\sum_{\substack{R(i)\\ -n\le i<0}} a_i\right)\left(\sum_{\substack{S(j)\\ 0\le j<n}} b_j + \sum_{\substack{S(j)\\ -n\le j<0}} b_j\right)\right)\\
&= \lim_{n\to\infty}\left(\sum_{\substack{R(i)\\ 0\le i<n}} a_i\left(\sum_{\substack{S(j)\\ 0\le j<n}} b_j + \sum_{\substack{S(j)\\ -n\le j<0}} b_j\right) + \sum_{\substack{R(i)\\ -n\le i<0}} a_i\left(\sum_{\substack{S(j)\\ 0\le j<n}} b_j + \sum_{\substack{S(j)\\ -n\le j<0}} b_j\right)\right)\\
&= \lim_{n\to\infty}\left(\sum_{\substack{R(i)\\ 0\le i<n}} \left(\sum_{\substack{S(j)\\ 0\le j<n}} a_i b_j + \sum_{\substack{S(j)\\ -n\le j<0}} a_i b_j\right) + \sum_{\substack{R(i)\\ -n\le i<0}} \left(\sum_{\substack{S(j)\\ 0\le j<n}} a_i b_j + \sum_{\substack{S(j)\\ -n\le j<0}} a_i b_j\right)\right)\\
&= \lim_{n\to\infty}\left(\sum_{\substack{R(i)\\ 0\le i<n}} \left(\sum_{S(j)} a_i b_j\right) + \sum_{\substack{R(i)\\ -n\le i<0}} \left(\sum_{S(j)} a_i b_j\right)\right)\\
&= \sum_{R(i)}\left(\sum_{S(j)} a_i b_j\right)
\end{aligned}$$

as we needed to show. ☐

6. [*HM20*] Prove that rule (d) is valid for an arbitrary infinite series, provided that any three of the four sums exist.

Proposition. *Manipulating the domain,* $\sum_{R(j)} a_j + \sum_{S(j)} a_j = \sum_{R(j)\vee S(j)} a_j + \sum_{R(j)\wedge S(j)} a_j$ *is valid for arbitrary infinite series, provided that any three of the four sums exist.*

Proof. Assume that R and S are arbitrary infinite series. We must show that if any three of the sums $\sum_{R(j)} a_j$, $\sum_{S(j)} a_j$, $\sum_{R(j)\vee S(j)} a_j$, $\sum_{R(j)\wedge S(j)} a_j$ exist, then we may manipulate the domain so that $\sum_{R(j)} a_j + \sum_{S(j)} a_j = \sum_{R(j)\vee S(j)} a_j + \sum_{R(j)\wedge S(j)} a_j$. It is sufficient to show that the convergence of three sums is sufficient for the convergence of the fourth.

Case 1. [$\sum_{R(j)} a_j$, $\sum_{S(j)} a_j$, $\sum_{R(j)\vee S(j)} a_j$ *converge.*] We must show that $\sum_{R(j)\wedge S(j)} a_j$ converges. But if the sums $\sum_{R(j)} a_j$, $\sum_{S(j)} a_j$ exist, then clearly their conjunction $\sum_{R(j)\wedge S(j)} a_j$ exists.

Case 2. [$\sum_{R(j)} a_j$, $\sum_{S(j)} a_j$, $\sum_{R(j)\wedge S(j)} a_j$ *converge.*] We must sow that $\sum_{R(j)\vee S(j)} a_j$ converges. But if the sums $\sum_{R(j)} a_j$, $\sum_{S(j)} a_j$ exist, then clearly their disjunction $\sum_{R(j)\vee S(j)} a_j$ exists.

Case 3. [$\sum_{R(j)} a_j$, $\sum_{R(j)\vee S(j)} a_j$, $\sum_{R(j)\wedge S(j)} a_j$ *converge.*] We must show that $\sum_{S(j)} a_j$ converges. But if the conjunction $\sum_{R(j)\wedge S(j)} a_j$ exists, then clearly the single sum $\sum_{S(j)} a_j$ exists.

Case 4. [$\sum_{S(j)} a_j$, $\sum_{R(j)\vee S(j)} a_j$, $\sum_{R(j)\wedge S(j)} a_j$ *converge.*] We must show that $\sum_{R(j)} a_j$ converges. But if the conjunction $\sum_{R(j)\wedge S(j)} a_j$ exists, then clearly the single sum $\sum_{R(j)} a_j$ exists.

Therefore, in all cases, we have shown that the convergence of three sums is sufficient for the convergence of the fourth. □

7. [*HM23*] Given that c is an integer, show that $\sum_{R(j)} a_j = \sum_{R(c-j)} a_{c-j}$, even if both series are infinite.

Proposition. $\sum_{R(j)} a_j = \sum_{R(c-j)} a_{c-j}$*, even if both series are infinite.*

Proof. Assume that R is an arbitrary relation. We must show that $\sum_{R(j)} a_j = \sum_{R(c-j)} a_{c-j}$, even if both series are infinite. Since $R'(j) = c - j$ is a permutation of the integers, proving the finite case, we must show the infinite case. But

$$\begin{aligned}
\sum_{R(j)} a_j &= \left(\lim_{n\to\infty} \sum_{\substack{R(j)\\ 0\le j<n}} a_j\right) + \left(\lim_{n\to\infty} \sum_{\substack{R(j)\\ -n\le j<0}} a_j\right)\\
&= \left(\lim_{n\to\infty} \sum_{\substack{R(c-j)\\ 0\le c-j<n}} a_{c-j}\right) + \left(\lim_{n\to\infty} \sum_{\substack{R(c-j)\\ -n\le c-j<0}} a_{c-j}\right)\\
&= \sum_{R(c-j)} a_{c-j}
\end{aligned}$$

as we needed to show. □

8. [*HM25*] Find an example of infinite series in which Eq. (7) is false.

An example of an infinite series in which $\sum_{R(i)} \sum_{S(j)} a_{ij} \neq \sum_{S(j)} \sum_{R(j)} a_{ij}$ is given by

$$a_{ij} = \begin{cases} \text{-1} & \text{if } i-1=j\\ 1 & \text{if } i+1=j\\ 0 & \text{otherwise} \end{cases}$$

for $S(i) \equiv i \geq 0$, $R(j) \equiv j \geq 0$. Then

$$\begin{aligned}
\sum_{R(i)} \sum_{S(j)} a_{ij} &= \lim_{n\to\infty} \sum_{0\leq i<n} \sum_{0\leq j<n} a_{ij} \\
&= \lim_{n\to\infty} \left(\sum_{0\leq i<1} \sum_{0\leq j<n} a_{ij} + \sum_{1\leq i<n} \sum_{0\leq j<n} a_{ij} \right) \\
&= \lim_{n\to\infty} \left(a_{01} + \sum_{1\leq i<n} \sum_{0\leq j<n} a_{ij} \right) \\
&= \lim_{n\to\infty} \left(-1 + \sum_{1\leq i<n} \sum_{0\leq j<n} a_{ij} \right) \\
&= -1
\end{aligned}$$

and

$$\begin{aligned}
\sum_{S(j)} \sum_{R(i)} a_{ij} &= \lim_{n\to\infty} \sum_{0\leq j<n} \sum_{0\leq i<n} a_{ij} \\
&= \lim_{n\to\infty} \left(\sum_{0\leq j<1} \sum_{0\leq i<n} a_{ij} + \sum_{1\leq j<n} \sum_{0\leq i<n} a_{ij} \right) \\
&= \lim_{n\to\infty} \left(a_{10} + \sum_{1\leq j<n} \sum_{0\leq i<n} a_{ij} \right) \\
&= \lim_{n\to\infty} \left(1 + \sum_{1\leq j<n} \sum_{0\leq i<n} a_{ij} \right) \\
&= 1.
\end{aligned}$$

▶ **9.** [*05*] Is the derivation of Eq. (14) valid even if $n = -1$?

No, the derivation for Eq. (14) is not valid if $n = -1$, since the application of rule (d) assumes $n \geq 0$. That is, if $n = -1$

$$\sum_{0\leq j\leq -1} ax^j \neq a + \sum_{1\leq j\leq -1} ax^j$$

(depsite the fact that $\sum_{0\leq j\leq -1} ax^j = 0 = a\left(\frac{1-x^{-1+1}}{1-x}\right)$).

10. [*05*] Is the derivation of Eq. (14) valid even if $n = -2$?

No, the derivation for Eq. (14) is not valid if $n = -2$, since the application of rule (d) assumes $n \geq 0$. That is, if $n = -2$

$$\sum_{0\leq j\leq -2} ax^j \neq a + \sum_{1\leq j\leq -2} ax^j.$$

11. [*03*] What should the right-hand side of Eq. (14) be if $x = 1$?

In the case $x = 1$

$$\begin{aligned}
\sum_{0\leq j\leq n} ax^j &= \sum_{0\leq j\leq n} a(1)^j \\
&= \sum_{0\leq j\leq n} a \\
&= (n+1)a.
\end{aligned}$$

12. [*10*] What is $1 + \frac{1}{7} + \frac{1}{49} + \frac{1}{343} + \cdots + (\frac{1}{7})^n$?

The sum $1 + \frac{1}{7} + \frac{1}{49} + \frac{1}{343} + \cdots + (\frac{1}{7})^n$ is simply a geometric progression whose closed form is given by Eq. (14) with $a = 1$ and $x = \frac{1}{7}$. This yields

$$\begin{aligned}\sum_{0\le j\le n} (1)\left(\frac{1}{7}\right)^j &= (1)\left(\frac{1-\left(\frac{1}{7}\right)^{n+1}}{1-\frac{1}{7}}\right)\\ &= \frac{1-\left(\frac{1}{7}\right)^{n+1}}{6/7}\\ &= \frac{7}{6}\left(1 - 1/7^{n+1}\right).\end{aligned}$$

13. [*10*] Using Eq. (15) and assuming that $m \le n$, evaluate $\sum_{j=m}^{n} j$.

Since

$$\sum_{m\le j\le n} j = \sum_{0\le j\le n} j - \sum_{0\le j\le m-1} j$$

we can use Eq. (15) with $a = 0$ and $b = 1$ to evaluate each term on the right-hand side as

$$\begin{aligned}\sum_{m\le j\le n} j &= \sum_{0\le j\le n} j - \sum_{0\le j\le m-1} j\\ &= \frac{1}{2}n(n+1) - \frac{1}{2}(m-1)m\\ &= \frac{1}{2}(n(n+1) - m(m-1)).\end{aligned}$$

14. [*11*] Using the result of the previous exercise, evaluate $\sum_{j=m}^{n}\sum_{k=r}^{s} jk$.

Since $\sum_{m\le j\le n} j = \frac{1}{2}(n(n+1) - m(m-1))$, we can evaluate the sum as

$$\begin{aligned}\sum_{m\le j\le n}\sum_{r\le k\le s} jk &= \left(\sum_{m\le j\le n} j\right)\left(\sum_{r\le k\le s} k\right)\\ &= \left(\frac{1}{2}(n(n+1) - m(m-1))\right)\left(\frac{1}{2}(s(s+1) - r(r-1))\right)\\ &= \frac{1}{4}\left((n(n+1) - m(m-1))(s(s+1) - r(r-1))\right).\end{aligned}$$

▶ **15.** [*M22*] Compute the sum $1 \times 2 + 2 \times 2^2 + 3 \times 2^3 + \cdots + n \times 2^n$ for small values of n. Do you see the pattern developing in these numbers? If not, discover it by manipulations similar to those leading up to Eq. (14).

We can manipulate the sum as

$$\begin{aligned}
\sum_{0\le k\le n} k2^k &= 0 + \sum_{1\le k\le n} k2^k \\
&= \sum_{1\le k\le n} k2^k \\
&= 2\sum_{1\le k\le n} k2^{k-1} \\
&= 2\sum_{0\le k\le n-1} (k+1)2^k \\
&= 2\left(\sum_{0\le k\le n-1} k2^k + \sum_{0\le k\le n-1} 2^k\right) \\
&= 2\left(\sum_{0\le k\le n} k2^k - n2^n + \sum_{0\le k\le n-1} 2^k\right) \\
&= 2\left(\sum_{0\le k\le n} k2^k - n2^n - (1-2^n)\right) \\
&= 2\sum_{0\le k\le n} k2^k - n2^{n+1} - (2-2^{n+1}).
\end{aligned}$$

Comparing the first with the last yields

$$\begin{aligned}
\sum_{0\le k\le n} k2^k &= n2^{n+1} + 2 - 2^{n+1} \\
&= 2(n2^n - 2^n + 1).
\end{aligned}$$

16. [*M22*] Prove that

$$\sum_{j=0}^{n} jx^j = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2},$$

if $x \neq 1$, without using mathematical induction.

Proposition. *$\sum_{0\le j\le n} jx^j = \frac{nx^{n+2}-(n+1)x^{n+1}+x}{(x-1)^2}$ if $x \neq 1$.*

Proof. Assume that $x \neq 1$, $n \geq 0$. Then

$$\begin{aligned}
\sum_{0 \le j \le n} jx^j &= 0 + \sum_{1 \le j \le n} jx^j \\
&= \sum_{1 \le j \le n} jx^j \\
&= x \sum_{1 \le j \le n} jx^{j-1} \\
&= x \sum_{0 \le j \le n-1} (j+1)x^j \\
&= x \left(\sum_{0 \le j \le n-1} jx^j + \sum_{0 \le j \le n-1} x^j \right) \\
&= x \left(\sum_{0 \le j \le n} jx^j - nx^n + \sum_{0 \le j \le n-1} x^j \right) \\
&= x \left(\sum_{0 \le j \le n} jx^j - nx^n + \frac{1-x^n}{1-x} \right) \\
&= x \sum_{0 \le j \le n} jx^j - nx^{n+1} + x\frac{1-x^n}{1-x}.
\end{aligned}$$

Comparing the first relation with the last, we have

$$(1-x) \sum_{0 \le j \le n} jx^j = -nx^{n+1} + x\frac{1-x^n}{1-x};$$

hence we obtain the formula

$$\begin{aligned}
\sum_{0 \le j \le n} jx^j &= \frac{-(1-x)nx^{n+1}}{(1-x)^2} + x\frac{1-x^n}{(1-x)^2} \\
&= \frac{-nx^{n+1} + nx^{n+2} + x - x^{n+1}}{(1-x)^2} \\
&= \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(1-x)^2}
\end{aligned}$$

as we needed to show. □

▶ **17.** [*M00*] Let S be a set of integers. What is $\sum_{j \in S} 1$?

$\sum_{j \in S} 1$ is the *cardinality*—the number of elements—of S; that is,

$$\sum_{j \in S} 1 = |S|.$$

18. [*M20*] Show how to interchange the order of summation as in Eq. (9) given that $R(i)$ is the relation "n is a multiple of i" and $S(i,j)$ is the relation "$1 \le j < i$."

In Eq. (9), we have

$$\sum_{R(i)} \sum_{S(i,j)} a_{ij} = \sum_{S'(j)} \sum_{R'(i,j)} a_{ij}$$

for $S'(j) \equiv (\exists i)(R(i) \wedge S(i,j))$ and $R'(i,j) \equiv R(i) \wedge S(i,j)$.

In the case that $R(i) \equiv (\exists k)(ki = n)$ and $S(i,j) \equiv 1 \le j < i$, we find

$$\begin{aligned} S'(j) &\equiv (\exists i)(R(i) \wedge S(i,j)) \\ &\equiv (\exists i)((\exists k)(ki = n) \wedge 1 \le j < i) \\ &\equiv 1 \le j < n && \text{for } i \le n,\ k \ge 1 \end{aligned}$$

and

$$\begin{aligned} R'(i,j) &\equiv R(i) \wedge S(i,j) \\ &\equiv (\exists k)(ki = n) \wedge 1 \le j < i \\ &\equiv (\exists k)(ki = n) \wedge j < i; \end{aligned}$$

that is, $S'(j)$ the relation "$1 \le j < n$" and $R'(i,j)$ the relation "n is a multiple of i and $i > j$."

19. [*20*] What is $\sum_{j=m}^{n}(a_j - a_{j-1})$?

We have that

$$\begin{aligned} \sum_{m \le j \le n}(a_j - a_{j-1}) &= \sum_{m \le j \le n} a_j - \sum_{m \le j \le n} a_{j-1} \\ &= \sum_{m \le j \le n} a_j - \sum_{m-1 \le j \le n-1} a_j \\ &= \sum_{m \le j \le n} a_j - a_{m-1} - \sum_{m \le j \le n} a_j + a_n \\ &= a_n - a_{m-1} \end{aligned}$$

assuming $m \le n$.

▶ **20.** [*25*] Dr. I. J. Matrix has observed a remarkable sequence of formulas:

$$9 \times 1 + 2 = 11, 9 \times 12 + 3 = 111, 9 \times 123 + 4 = 1111, 9 \times 1234 + 5 = 11111.$$

a. Write the good doctor's great discovery in terms of the $\sum$-notation.

b. Your answer to part (a) undoubtedly involves the number 10 as a base of the decimal system; generalize this formula so that you get a formula that will perhaps work in any base b.

c. Prove your formula from part (b) by using formulas derived in the text or in exercise 16 above.

The remarkable sequence of formulas observed by Dr. I. J. Matrix are analyzed below.

a. The good doctor's great discovery in terms of the $\sum$-notation is

$$(10-1)\sum_{0 \le k \le n}(n-k)10^k + n + 1 = \sum_{0 \le k \le n} 10^k.$$

b. The above formula may be generalized for perhaps any base b as

$$(b-1)\sum_{0 \le k \le n}(n-k)b^k + n + 1 = \sum_{0 \le k \le n} b^k.$$

c. We may prove the above formula.

Proposition. $(b-1)\sum_{0 \le k \le n}(n-k)b^k + n + 1 = \sum_{0 \le k \le n} b^k$.

Proof. Assume an arbitrary base $b \geq 2$, and $n \geq 0$. We must show that

$$(b-1)\sum_{0\leq k\leq n}(n-k)b^k + n + 1 = \sum_{0\leq k\leq n} b^k.$$

But

$$\begin{aligned}
&(b-1)\sum_{0\leq k\leq n}(n-k)b^k + n + 1\\
&\quad = (b-1)\left(n\sum_{0\leq k\leq n} b^k - \sum_{0\leq k\leq n} kb^k\right) + n + 1\\
&\quad = (b-1)\left(n\left(\frac{1-b^{n+1}}{1-b}\right) - \sum_{0\leq k\leq n} kb^k\right) + n + 1 && \text{by Eq. (14)}\\
&\quad = (b-1)\left(n\left(\frac{1-b^{n+1}}{1-b}\right) - \left(\frac{nb^{n+2}-(n+1)b^{n+1}+b}{(b-1)^2}\right)\right) + n + 1 && \text{by exercise 16}\\
&\quad = n(b^{n+1}-1) + \frac{b(b^n(-nb+n+1)-1)}{b-1} + n + 1\\
&\quad = \frac{b^{n+1}+n-b(n+1)}{b-1} + n + 1\\
&\quad = \frac{1-b^{n+1}}{1-b}\\
&\quad = \sum_{0\leq k\leq n} b^k && \text{by Eq. (14)}
\end{aligned}$$

as we needed to show. □

▶ **21.** [*M25*] Derive rule (d) from (8) and (17).

We may derive rule (d) for manipulating the domain

$$\sum_{R(j)} a_j + \sum_{S(j)} a_j = \sum_{R(j)\vee S(j)} a_j + \sum_{R(j)\wedge S(j)} a_j$$

from Eq. (8)

$$\sum_{R(i)}(b_i + c_i) = \sum_{R(i)} b_i + \sum_{R(i)} c_i$$

and Eq. (17)

$$\sum_{R(j)} a_j = \sum_j a_j[R(j)].$$

Given that $[p]+[q] = [p\vee q] + [p\wedge q]$, as evidenced by the following table

$[p]$	$[q]$	$[p]+[q]$	$[p\vee q]$	$[p\wedge q]$	$[p\vee q]+[p\wedge q]$
0	0	0	0	0	0
0	1	1	1	0	1
1	0	1	1	0	1
1	1	2	1	1	2

we have

$$\begin{aligned}
\sum_{R(j)} a_j + \sum_{S(j)} a_j &= \sum_j a_j[R(j)] + \sum_j a_j[S(j)] && \text{by Eq. (17)}\\
&= \sum_j (a_j[R(j)] + a_j[S(j)]) && \text{by Eq. (8)}\\
&= \sum_j a_j([R(j)] + [S(j)])\\
&= \sum_j a_j([R(j) \vee S(j)] + [R(j) \wedge S(j)])\\
&= \sum_j a_j[R(j) \vee S(j)] + \sum_j a_j[R(j) \wedge S(j)] && \text{by Eq. (8)}\\
&= \sum_{R(j)\vee S(j)} a_j + \sum_{R(j)\wedge S(j)} a_j. && \text{by Eq. (17)}
\end{aligned}$$

► **22.** [*20*] State the appropriate analogs of Eqs. (5), (7), (8), and (11) for *products* instead of sums.

We have the following analogs for products: *change of variable*:

$$\prod_{R(i)} a_i = \prod_{R(j)} = \prod_{R(p(j))} a_{p(j)};$$

interchanging order of production:

$$\prod_{R(i)} \prod_{S(j)} a_{ij} = \prod_{S(j)} \prod_{R(i)} a_{ij};$$

a special case of the above:

$$\prod_{R(i)} (b_i c_i) = \left(\prod_{R(i)} b_i\right)\left(\prod_{R(i)} c_i\right);$$

and *manipulating the domain*:

$$\left(\prod_{R(j)} a_j\right)\left(\prod_{S(j)} a_j\right) = \left(\prod_{R(j)\vee S(j)} a_j\right)\left(\prod_{R(j)\wedge S(j)} a_j\right).$$

23. [*10*] Explain why it is a good idea to define $\sum_{R(j)} a_j$ and $\prod_{R(j)} a_j$ as zero and one, respectively, when no integers satisify $R(j)$.

It is a good idea to define $\sum_{j\in\emptyset} a_j = 0$ and $\prod_{j\in\emptyset} a_j = 1$ as they are the identity elements for the operations of addition and multiplication, respectively. This way, $\sum_{j\in\emptyset} a_j + \sum_{R(j)} a_j = \sum_{R(j)} a_j$ and $\left(\prod_{j\in\emptyset} a_j\right)\left(\prod_{R(j)} a_j\right) = \prod_{R(j)} a_j$.

24. [*20*] Suppose that $R(j)$ is true for only finitely many j. By induction on the number of integers satisfying $R(j)$, prove that $\log_b \prod_{R(j)} a_j = \sum_{R(j)}(\log_b a_j)$, assuming that all $a_j > 0$.

Proposition. $\log_b \prod_{R(j)} a_j = \sum_{R(j)}(\log_b a_j)$ *for all* $a_j > 0$.

Proof. Suppose $a_j > 0$ for $0 \le j \le n$. We must show that $\log_b \prod_{0\le j\le n} a_j = \sum_{0\le j\le n}(\log_b a_j)$.

If $n = 0$, clearly $\log_b \prod_{0\le j\le 0} a_j = \log_b a_0 = \sum_{0\le j\le 0}(\log_b a_j)$. Then, assuming

$$\log_b \prod_{0\le j\le k} a_j = \sum_{0\le j\le k} (\log_b a_j),$$

we must show that

$$\log_b \prod_{0\le j\le k+1} a_j = \sum_{0\le j\le k+1} (\log_b a_j).$$

But

$$\begin{aligned}\log_b \prod_{0\le j\le k+1} a_j &= \log_b \left(a_{k+1} \prod_{0\le j\le k} a_j \right)\\ &= \log_b \prod_{0\le j\le k} a_j + \log_b a_{k+1}\\ &= \sum_{0\le j\le k} (\log_b a_j) + \log_b a_{k+1}\\ &= \sum_{0\le j\le k+1} (\log_b a_j)\end{aligned}$$

as we needed to show. □

▶ **25.** [*15*] Consider the following derivation; is anything amiss?

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{n} \frac{1}{a_j}\right) = \sum_{1\le i\le n}\sum_{1\le j\le n} \frac{a_i}{a_j} = \sum_{1\le i\le n}\sum_{1\le i\le n} \frac{a_i}{a_i} = \sum_{i=1}^{n} 1 = n.$$

Yes. For one,

$$\sum_{1\le i\le n}\sum_{1\le j\le n} \frac{a_i}{a_j} \ne \sum_{1\le i\le n}\sum_{1\le i\le n} \frac{a_i}{a_i};$$

and for another,

$$\sum_{1\le i\le n}\sum_{1\le i\le n} \frac{a_i}{a_i} \ne \sum_{i=1}^{n} 1$$

(in fact, $\sum_{1\le i\le n}\sum_{1\le i\le n} \frac{a_i}{a_i} = \sum_{i=1}^{n} n = n^2$).

26. [*25*] Show that $\prod_{i=0}^{n}\prod_{j=0}^{i} a_i a_j$ may be expressed in terms of $\prod_{i=0}^{n} a_i$ by manipulating the $\prod$-notation as stated in exercise 22.

Proposition. $\prod_{0\le i\le n}\prod_{0\le j\le i} a_i a_j = \left(\prod_{0\le i\le n} a_i\right)^{n+2}$.

Proof. We must show that $\prod_{0\le i\le n}\prod_{0\le j\le i} a_i a_j = \left(\prod_{0\le i\le n} a_i\right)^{n+2}$.

First note that

$$\prod_{0\le i\le n}\prod_{0\le j\le i} a_i a_j = \prod_{0\le i\le n}\prod_{i\le j\le n} a_i a_j;$$

and then that

$$\begin{aligned}
\left(\prod_{0\le i\le n}\prod_{0\le j\le i} a_i a_j\right)^2 &= \left(\prod_{0\le i\le n}\prod_{0\le j\le i} a_i a_j\right)\left(\prod_{0\le i\le n}\prod_{0\le j\le i} a_i a_j\right)\\
&= \prod_{0\le i\le n}\left(\left(\prod_{0\le j\le i} a_i a_j\right)\left(\prod_{0\le j\le i} a_i a_j\right)\right)\\
&= \prod_{0\le i\le n}\left(\left(\prod_{0\le j\le i} a_i a_j\right)\left(\prod_{i\le j\le n} a_i a_j\right)\right)\\
&= \prod_{0\le i\le n}\left(\left(\prod_{0\le j\le n} a_i a_j\right)(a_i a_i)\right)\\
&= \left(\prod_{0\le i\le n}\prod_{0\le j\le n} a_i a_j\right)\left(\prod_{0\le i\le n} a_i^2\right)\\
&= \left(\prod_{0\le i\le n}\left(a_i^{n+1}\prod_{0\le j\le n} a_j\right)\right)\left(\prod_{0\le i\le n} a_i\right)^2\\
&= \left(\prod_{0\le i\le n} a_i\right)^{n+1}\left(\prod_{0\le i\le n}\prod_{0\le j\le n} a_j\right)\left(\prod_{0\le i\le n} a_i\right)^2\\
&= \left(\prod_{0\le i\le n} a_i\right)^{n+1}\left(\prod_{0\le j\le n} a_j\right)^{n+1}\left(\prod_{0\le i\le n} a_i\right)^2\\
&= \left(\prod_{0\le i\le n} a_i\right)^{2n+4}.
\end{aligned}$$

Therefore,

$$\prod_{0\le i\le n}\prod_{0\le j\le i} a_i a_j = \left(\prod_{0\le i\le n} a_i\right)^{n+2}$$

as we needed to show. □

27. [*M20*] Generalize the result of exercise 1.2.1-9 by proving that

$$\prod_{j=1}^{n}(1-a_j) \ge 1-\sum_{j=1}^{n} a_j,$$

assuming that $0 < a_j < 1$.

Proposition. $\prod_{1\le j\le n}(1-a_j) \ge 1-\sum_{1\le j\le n} a_j$ *if* $0 < a_j < 1$.

Proof. Let $0 < a_j < 1$ for $1 \le j \le n$, $n \ge 0$. We must show that

$$\prod_{1\le j\le n}(1-a_j) \ge 1-\sum_{1\le j\le n} a_j.$$

If $n = 0$, then clearly $\prod_{1\le j\le n}(1-a_j) = 1 \ge 1 = 1-\sum_{1\le j\le n} a_j$. Then, assuming

$$\prod_{1\le j\le k}(1-a_j) \ge 1-\sum_{1\le j\le k} a_j,$$

we must show that

$$\prod_{1\le j\le k+1}(1-a_j)\ge 1-\sum_{1\le j\le k+1}a_j.$$

But since $a_{k+1}\sum_{0\le j\le k}a_k\ge 1$,

$$\begin{aligned}\prod_{1\le j\le k+1}(1-a_j)&=\left(\prod_{1\le j\le k}(1-a_j)\right)(1-a_{k+1})\\&\ge\left(1-\sum_{1\le j\le k}a_j\right)(1-a_{k+1})\\&=\left(1-\sum_{1\le j\le k}a_j\right)-\left(a_{k+1}-a_{k+1}\sum_{0\le j\le k}a_j\right)\\&=1-\sum_{0\le j\le k}a_j-a_{k+1}+a_{k+1}\sum_{0\le j\le k}a_k\\&\ge 1-\left(\sum_{0\le j\le k}a_j+a_{k+1}\right)\\&=1-\sum_{1\le j\le k+1}a_j\end{aligned}$$

as we needed to show. □

28. [*M22*] Find a simple formula for $\prod_{j=2}^{n}(1-1/j^2)$.

We find that

$$\begin{aligned}\prod_{2\le j\le n}(1-\frac{1}{j^2})&=\prod_{2\le j\le n}\frac{j^2-1}{j^2}\\&=\left(\prod_{2\le j\le n}\frac{1}{j^2}\right)\left(\prod_{2\le j\le n}(j-1)(j+1)\right)\\&=\left(\prod_{2\le j\le n}\frac{1}{j}\right)^2\left(\prod_{1\le j\le n-1}j\right)\left(\prod_{3\le j\le n+1}j\right)\\&=\left(\frac{1}{n!}\right)^2(n-1)!\frac{(n+1)!}{2}\\&=\frac{n+1}{2n}.\end{aligned}$$

► **29.** [*M30*] (a) Express $\sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{k=0}^{j}a_ia_ja_k$ in terms of the multiple-sum notation explained at the end of the section. (b) Express the same sum in terms of $\sum_{i=0}^{n}a_i$, $\sum_{i=0}^{n}a_i^2$, and $\sum_{i=0}^{n}a_i^3$ [see Eq. (13)].

a. We can express the sum in terms of the multiple-sum notation as

$$\sum_{0\le i\le n}\sum_{0\le j\le i}\sum_{0\le k\le j}a_ia_ja_k=\sum_{0\le k\le j\le i\le n}a_ia_ja_k.$$

b. From Eq. (13) for arbitrary $i>0$ we have

$$\sum_{0\le j\le i-1}\sum_{0\le k\le j}a_ja_k=\frac{1}{2}\left(\left(\sum_{0\le j\le i-1}a_j\right)^2+\sum_{0\le j\le i-1}a_j^2\right)$$

or equivalently

$$\left(\sum_{0\le j\le i-1} a_j\right)^2 = 2\sum_{0\le j\le i-1}\sum_{0\le k\le j} a_j a_k - \sum_{0\le j\le i-1} a_j^2.$$

Also for arbitrary $i > 0$ we have that

$$\begin{aligned}
&\left(\sum_{0\le j\le i} a_j\right)^3 \\
&= \left(\sum_{0\le j\le i-1} a_j + a_i\right)^3 \\
&= \left(\sum_{0\le j\le i-1} a_j\right)^3 + 3a_i\left(\sum_{0\le j\le i-1} a_j\right)^2 + 3a_i^2\sum_{0\le j\le i-1} a_j + a_i^3 \\
&= \left(\sum_{0\le j\le i-1} a_j\right)^3 + 3a_i\left(2\sum_{0\le j\le i-1}\sum_{0\le k\le j} a_j a_k - \sum_{0\le j\le i-1} a_j^2\right) + 3a_i^2\sum_{0\le j\le i-1} a_j + a_i^3 \\
&= \left(\sum_{0\le j\le i-1} a_j\right)^3 + 6\sum_{0\le j\le i-1}\sum_{0\le k\le j} a_i a_j a_k - 3\sum_{0\le j\le i-1} a_i a_j^2 + 3\sum_{0\le j\le i-1} a_i^2 a_j + a_i^3 \\
&= \left(\sum_{0\le j\le i-1} a_j\right)^3 + 6\sum_{0\le j\le i-1}\sum_{0\le k\le j} a_i a_j a_k - 3\sum_{0\le j\le i} a_i a_j^2 + 3\sum_{0\le j\le i} a_i^2 a_j + a_i^3 \\
&= \left(\sum_{0\le j\le i-1} a_j\right)^3 + 6\sum_{0\le j\le i}\sum_{0\le k\le j} a_i a_j a_k - 6\sum_{0\le j\le i} a_i^2 a_j - 3\sum_{0\le j\le i} a_i a_j^2 + 3\sum_{0\le j\le i} a_i^2 a_j + a_i^3 \\
&= \left(\sum_{0\le j\le i-1} a_j\right)^3 + 6\sum_{0\le j\le i}\sum_{0\le k\le j} a_i a_j a_k - 3\sum_{0\le j\le i} a_i a_j^2 - 3\sum_{0\le j\le i} a_i^2 a_j + a_i^3 \\
&= \left(\sum_{0\le j\le i-1} a_j\right)^3 + 6\sum_{0\le j\le i}\sum_{0\le k\le j} a_i a_j a_k - 3\sum_{0\le j\le i} a_i a_j(a_i + a_j) + a_i^3
\end{aligned}$$

and in the trivial case for $i = 0$ that

$$\left(\sum_{0\le j\le i} a_j\right)^3 = a_0^3 = (0)^3 + 6a_0^3 - 3(2)a_0^3 + a_0^3$$

so that

$$6\sum_{0\le j\le i}\sum_{0\le k\le j} a_i a_j a_k = \left(\sum_{0\le j\le i} a_j\right)^3 - \left(\sum_{0\le j\le i-1} a_j\right)^3 + 3\sum_{0\le j\le i} a_i a_j(a_i + a_j) - a_i^3$$

and so that for $0 \le i \le n$ we have $6\sum_{0\le i\le n}\sum_{0\le j\le i}\sum_{0\le k\le j} a_i a_j a_k$ equivalent to

$$\sum_{0\le i\le n}\left(\left(\sum_{0\le j\le i} a_j\right)^3 - \left(\sum_{0\le j\le i-1} a_j\right)^3\right) + 3\sum_{0\le i\le n}\sum_{0\le j\le i} a_i a_j(a_i + a_j) - \sum_{0\le i\le n} a_i^3.$$

We may also prove by induction that

$$\sum_{0\le i\le n}\left(\left(\sum_{0\le j\le i} a_j\right)^3-\left(\sum_{0\le j\le i-1} a_j\right)^3\right)=\left(\sum_{0\le i\le n} a_i\right)^3.$$

If $n=0$, clearly $a_0^3-(0)^3=a_0^3$. Then, assuming

$$\sum_{0\le i\le k}\left(\left(\sum_{0\le j\le i} a_j\right)^3-\left(\sum_{0\le j\le i-1} a_j\right)^3\right)=\left(\sum_{0\le i\le k} a_i\right)^3$$

we must show that

$$\sum_{0\le i\le k+1}\left(\left(\sum_{0\le j\le i} a_j\right)^3-\left(\sum_{0\le j\le i-1} a_j\right)^3\right)=\left(\sum_{0\le i\le k+1} a_i\right)^3.$$

But

$$\begin{aligned}
&\sum_{0\le i\le k+1}\left(\left(\sum_{0\le j\le i} a_j\right)^3-\left(\sum_{0\le j\le i-1} a_j\right)^3\right)\\
&=\sum_{0\le i\le k}\left(\left(\sum_{0\le j\le i} a_j\right)^3-\left(\sum_{0\le j\le i-1} a_j\right)^3\right)+\left(\sum_{0\le j\le k+1} a_j\right)^3-\left(\sum_{0\le j\le k} a_j\right)^3\\
&=\left(\sum_{0\le i\le k} a_i\right)^3+\left(\sum_{0\le j\le k+1} a_j\right)^3-\left(\sum_{0\le j\le k} a_j\right)^3\\
&=\left(\sum_{0\le j\le k+1} a_j\right)^3
\end{aligned}$$

And so finally, we have that

$$6\sum_{0\le i\le n}\sum_{0\le j\le i}\sum_{0\le k\le j} a_ia_ja_k=\left(\sum_{0\le i\le n} a_i\right)^3+3\sum_{0\le i\le n}\sum_{0\le j\le i} a_ia_j(a_i+a_j)-\sum_{0\le i\le n} a_i^3$$

and

$$
\begin{aligned}
&6\sum_{0\le i\le n}\sum_{0\le j\le i}\sum_{0\le k\le j} a_i a_j a_k \\
&\quad= \left(\sum_{0\le i\le n} a_i\right)^3 + 3\sum_{0\le i\le n}\sum_{0\le j\le i} a_i a_j(a_i+a_j) - \sum_{0\le i\le n} a_i^3 \\
&\quad= \left(\sum_{0\le i\le n} a_i\right)^3 + \sum_{0\le i\le n}\sum_{0\le j\le i} a_i a_j(a_i+a_j) - \sum_{0\le i\le n} a_i^3 + 2\sum_{0\le i\le n}\sum_{0\le j\le i} a_i a_j(a_i+a_j) \\
&\quad= \left(\sum_{0\le i\le n} a_i\right)^3 + \sum_{0\le i\le n}\left(\frac{1}{2}\sum_{0\le j\le i} a_i a_j(a_i+a_j) + \frac{1}{2}\sum_{i\le j\le n} a_i a_j(a_i+a_j)\right) \\
&\qquad - \sum_{0\le i\le n} a_i^3 + 2\sum_{0\le i\le n}\sum_{0\le j\le i} a_i a_j(a_i+a_j) \\
&\quad= \left(\sum_{0\le i\le n} a_i\right)^3 + \sum_{0\le i\le n}\left(\frac{1}{2}\sum_{0\le j\le n} a_i a_j^2 + \frac{1}{2}\sum_{0\le j\le n} a_i^2 a_j + a_i^3\right) \\
&\qquad - \sum_{0\le i\le n} a_i^3 + 2\sum_{0\le i\le n}\sum_{0\le j\le i} a_i a_j(a_i+a_j) \\
&\quad= \left(\sum_{0\le i\le n} a_i\right)^3 + \left(\sum_{0\le i\le n} a_i\right)\left(\sum_{0\le i\le n} a_i^2\right) + \sum_{0\le i\le n} a_i^3 - \sum_{0\le i\le n} a_i^3 + 2\sum_{0\le i\le n}\sum_{0\le j\le i} a_i a_j(a_i+a_j) \\
&\quad= \left(\sum_{0\le i\le n} a_i\right)^3 + \left(\sum_{0\le i\le n} a_i\right)\left(\sum_{0\le i\le n} a_i^2\right) + 2\sum_{0\le i\le n}\sum_{0\le j\le i} a_i a_j(a_i+a_j) \\
&\quad= \left(\sum_{0\le i\le n} a_i\right)^3 + \left(\sum_{0\le i\le n} a_i\right)\left(\sum_{0\le i\le n} a_i^2\right) + \sum_{0\le i\le n}\left(\sum_{0\le j\le i} a_i a_j(a_i+a_j) + \sum_{i\le j\le n} a_i a_j(a_i+a_j)\right) \\
&\quad= \left(\sum_{0\le i\le n} a_i\right)^3 + \left(\sum_{0\le i\le n} a_i\right)\left(\sum_{0\le i\le n} a_i^2\right) + \sum_{0\le i\le n}\left(\sum_{0\le j\le n} a_i a_j(a_i+a_j) + 2a_i^3\right) \\
&\quad= \left(\sum_{0\le i\le n} a_i\right)^3 + \left(\sum_{0\le i\le n} a_i\right)\left(\sum_{0\le i\le n} a_i^2\right) + \sum_{0\le i\le n}\sum_{0\le j\le n} a_i a_j^2 + \sum_{0\le i\le n}\sum_{0\le j\le n} a_i^2 a_j + 2\sum_{0\le i\le n} a_i^3 \\
&\quad= \left(\sum_{0\le i\le n} a_i\right)^3 + \left(\sum_{0\le i\le n} a_i\right)\left(\sum_{0\le i\le n} a_i^2\right) + 2\left(\sum_{0\le i\le n} a_i\right)\left(\sum_{0\le i\le n} a_i^2\right) + 2\sum_{0\le i\le n} a_i^3 \\
&\quad= \left(\sum_{0\le i\le n} a_i\right)^3 + 3\left(\sum_{0\le i\le n} a_i\right)\left(\sum_{0\le i\le n} a_i^2\right) + 2\sum_{0\le i\le n} a_i^3
\end{aligned}
$$

Therefore,

$$
\sum_{0\le i\le n}\sum_{0\le j\le i}\sum_{0\le k\le j} a_i a_j a_k = \frac{1}{6}\left(\sum_{0\le i\le n} a_i\right)^3 + \frac{1}{2}\left(\sum_{0\le i\le n} a_i\right)\left(\sum_{0\le i\le n} a_i^2\right) + \frac{1}{3}\sum_{0\le i\le n} a_i^3.
$$

▶ **30.** [*M23*] (J. Binet, 1812.) Without using induction, prove the identity

$$\left(\sum_{j=1}^{n} a_j x_j\right)\left(\sum_{j=1}^{n} b_j y_j\right) = \left(\sum_{j=1}^{n} a_j y_j\right)\left(\sum_{j=1}^{n} b_j x_j\right) + \sum_{1\le j\le k\le n}(a_j b_k - a_k b_j)(x_j y_k - x_k y_j).$$

[An important special case arises when $w_1, \ldots, w_n, z_1, \ldots, z_n$ are aribtrary complex numbers and we set $a_j = w_j$, $b_j = \bar{z}_j$, $x_j = \bar{w}_j$, $y_j = z_j$:

$$\left(\sum_{j=1}^{n}|w_j|^2\right)\left(\sum_{j=1}^{n}|z_j|^2\right) = \left|\sum_{j=1}^{n} w_j z_j\right|^2 + \sum_{1\le j\le k\le n}|w_j\bar{z}_k - w_k\bar{z}_j|^2.$$

The terms $|w_j\bar{z}_j|^2$ are nonnegative, so the famous *Cauchy-Schwarz inequality*

$$\left(\sum_{j=1}^{n}|w_j|^2\right)\left(\sum_{j=1}^{n}|z_j|^2\right) \ge \left|\sum_{j=1}^{n} w_j z_j\right|^2$$

is a consequence of Binet's formula.]

Proposition. $\left(\sum_{1\le j\le n} a_j x_j\right)\left(\sum_{1\le j\le n} b_j y_j\right) = \left(\sum_{1\le j\le n} a_j y_j\right)\left(\sum_{1\le j\le n} b_j x_j\right) + \sum_{1\le j\le n}\sum_{j<k\le n}(a_j b_k - a_k b_j)(x_j y_k - x_k y_j)$.

Proof. We need to show that

$$\left(\sum_{1\le j\le n} a_j x_j\right)\left(\sum_{1\le j\le n} b_j y_j\right) = \left(\sum_{1\le j\le n} a_j y_j\right)\left(\sum_{1\le j\le n} b_j x_j\right) + \sum_{1\le j\le n}\sum_{j<k\le n}(a_j b_k - a_k b_j)(x_j y_k - x_k y_j).$$

But

$$\begin{aligned}
&\sum_{1\le j\le n}\sum_{j<k\le n}(a_j b_k - a_k b_j)(x_j y_k - x_k y_j)\\
&\quad= \sum_{1\le j\le n}\sum_{j<k\le n} a_j b_k(x_j y_k - x_k y_j) - \sum_{1\le j\le n}\sum_{j<k\le n} a_k b_j(x_j y_k - x_k y_j)\\
&\quad= \sum_{1\le j\le n}\sum_{j<k\le n} a_j b_k(x_j y_k - x_k y_j) + \sum_{1\le j\le n}\sum_{j<k\le n} a_k b_j(x_k y_j - x_j y_k)\\
&\quad= \sum_{1\le j\le n}\sum_{j<k\le n} a_j b_k(x_j y_k - x_k y_j) + \sum_{1\le k\le n}\sum_{k<j\le n} a_j b_k(x_j y_k - x_k y_j)\\
&\quad= \sum_{1\le j\le n}\sum_{j<k\le n} a_j b_k(x_j y_k - x_k y_j) + \sum_{1\le j\le n}\sum_{1\le k<j} a_j b_k(x_j y_k - x_k y_j)\\
&\quad= \sum_{1\le j\le n}\sum_{1\le k<j} a_j b_k(x_j y_k - x_k y_j) + 0 + \sum_{1\le j\le n}\sum_{j<k\le n} a_j b_k(x_j y_k - x_k y_j)\\
&\quad= \sum_{1\le j\le n}\sum_{1\le k<j} a_j b_k(x_j y_k - y_j x_k) + \sum_{1\le j\le n} a_j b_j(x_j y_j - y_j x_j) + \sum_{1\le j\le n}\sum_{j<k\le n} a_j b_k(x_j y_k - y_j x_k)\\
&\quad= \sum_{1\le j\le n}\sum_{1\le k\le n} a_j b_k(x_j y_k - y_j x_k)\\
&\quad= \sum_{1\le j\le n}\sum_{1\le k\le n} a_j x_j b_k y_k - \sum_{1\le j\le n}\sum_{1\le k\le n} a_j y_j b_k x_k\\
&\quad= \left(\sum_{1\le j\le n} a_j x_j\right)\left(\sum_{1\le j\le n} b_j y_j\right) - \left(\sum_{1\le j\le n} a_j y_j\right)\left(\sum_{1\le j\le n} b_j x_j\right)
\end{aligned}$$

as we needed to show. □

31. [*M20*] Use Binet's formula to express the sum $\sum_{1\le j\le k\le n}(u_j - u_k)(v_j - v_k)$ in terms of $\sum_{j=1}^n u_j v_j$, $\sum_{j=1}^n u_j$, and $\sum_{j=1}^n v_j$.

We want to find an expression for

$$\sum_{1\le jn}\sum_{j<k\le n}(u_j - u_k)(v_j - v_k)$$

in terms of $\sum_{1\le j\le n} u_j v_j$, $\sum_{1\le j\le n} u_j$, and $\sum_{1\le j\le n} v_j$.

From Binet's formula we have that

$$\sum_{1\le j\le n}\sum_{j<k\le n}(a_j b_k - a_k b_j)(x_j y_k - x_k y_j) = \left(\sum_{1\le j\le n} a_j x_j\right)\left(\sum_{1\le j\le n} b_j y_j\right) - \left(\sum_{1\le j\le n} a_j y_j\right)\left(\sum_{1\le j\le n} b_j x_j\right).$$

If we let $a_j = u_j$, $x_j = v_j$, and $b_j = y_j = 1$, we find

$$\begin{aligned}
&\sum_{1\le j\le n}\sum_{j<k\le n}(u_j - u_k)(v_j - v_k)\\
&= \left(\sum_{1\le j\le n} u_j v_j\right)\left(\sum_{1\le j\le n} 1\right) - \left(\sum_{1\le j\le n} u_j\right)\left(\sum_{1\le j\le n} v_j\right)\\
&= n\sum_{1\le j\le n} u_j v_j - \left(\sum_{1\le j\le n} u_j\right)\left(\sum_{1\le j\le n} v_j\right).
\end{aligned}$$

[See *Soobschch. Mat. Obschch. Khar'kovskom Univ.* **4**, 2 (1882), 93–98.]

32. [*M20*] Prove that

$$\prod_{j=1}^{n}\sum_{i=1}^{m} a_{ij} = \sum_{1\le i_1,\ldots,i_n\le m} a_{i_1 1}\ldots a_{i_n n}.$$

Proposition. $\prod_{1\le j\le n}\sum_{1\le i_j\le m} a_{i_j j} = \sum_{1\le i_1,\ldots,i_n\le m} a_{i_1 1}\ldots a_{i_n n}$.

Proof. We need to show that

$$\prod_{1\le j\le n}\sum_{1\le i_j\le m} a_{i_j j} = \sum_{1\le i_1,\ldots,i_n\le m} a_{i_1 1}\ldots a_{i_n n}.$$

If $n = 1$, clearly $\sum_{1\le i_1\le m} a_{i_1 1} = \sum_{1\le i_1\le m} a_{i_1 1}$. Then, assuming that

$$\prod_{1\le j\le k}\sum_{1\le i_j\le m} a_{i_j j} = \sum_{1\le i_1,\ldots,i_k\le m} a_{i_1 1}\ldots a_{i_k k}$$

we must show that

$$\prod_{1\le j\le k+1}\sum_{1\le i_j\le m} a_{i_j j} = \sum_{1\le i_1,\ldots,i_{k+1}\le m} a_{i_1 1}\ldots a_{i_{k+1}(k+1)}.$$

But

$$\prod_{1\le j\le k+1}\sum_{1\le i_j\le m} a_{i_j j} = \left(\prod_{1\le j\le k}\sum_{1\le i_j\le m} a_{i_j j}\right)\sum_{1\le i_{k+1}\le m} a_{i_{k+1}(k+1)}$$
$$= \left(\sum_{1\le i_1,\ldots,i_k\le m} a_{i_1 1}\ldots a_{i_k k}\right)\sum_{1\le i_{k+1}\le m} a_{i_{k+1}(k+1)}$$
$$= \sum_{1\le i_1,\ldots,i_{k+1}\le m} a_{i_1 1}\ldots a_{i_{k+1}(k+1)}$$

as we needed to show. □

▶ **33.** [*M30*] One evening Dr. Matrix discovered some formulas that might even be classed as more remarkable than those of exercise 20:

$$\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}=0,$$
$$\frac{a}{(a-b)(a-c)}+\frac{b}{(b-a)(b-c)}+\frac{c}{(c-a)(c-b)}=0,$$
$$\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-a)(b-c)}+\frac{c^2}{(c-a)(c-b)}=1,$$
$$\frac{a^3}{(a-b)(a-c)}+\frac{b^3}{(b-a)(b-c)}+\frac{c^3}{(c-a)(c-b)}=a+b+c.$$

Prove that these formulas are a special case of a general law; let $x_1, x_2, \ldots, x_n$ be distinct numbers, and show that

$$\sum_{j=1}^{n}\left(x_j^r \Big/ \prod_{\substack{1\le k\le n\\ k\ne j}}(x_j-x_k)\right) = \begin{cases} 0, & \text{if } 0\le r<n-1,\\ 1, & \text{if } r=n-1;\\ \sum_{j=1}^n x_j, & \text{if } r=n.\end{cases}$$

Proposition. $\sum_{1\le i\le n}\left(\frac{x_i^r}{\prod_{\substack{1\le j\le n\\ j\ne i}}(x_i-x_j)}\right) = \begin{cases} 0 & \text{if } 0\le r<n-1\\ 1 & \text{if } r=n-1\\ \sum_{1\le i\le n} x_i & \text{if } r=n\end{cases}$ *if $x_1, x_2, \ldots, x_n$ distinct.*

Proof. For an arbitrary series of distinct numbers x_i, $1\le i\le n$, and for an arbitrary ι, $1\le\iota\le n$, let $P(x_\iota)=x_\iota^r$ for $0\le r\le n$, and $Q(x_\iota)=\prod_{\substack{1\le i\le n\\ i\ne\iota}}(x_\iota-x_i)$. By the fundamental theorem of algebra and the method of partial fractions, since $r=\deg P\le \deg Q+1=n$, we have that

$$\frac{P(x_\iota)}{Q(x_\iota)}=\frac{x_\iota^r}{\prod_{\substack{1\le i\le n\\ i\ne\iota}}(x_\iota-x_i)}=D(x_\iota)+\sum_{\substack{1\le i\le n\\ i\ne\iota}}\frac{c_i}{x_\iota-x_i}$$

for constants c_i, where $D(x_\iota)$ is the polynomial divisor with remainder $R(x_\iota)$ such that

$$P(x_\iota)=D(x_\iota)Q(x_\iota)+R(x_\iota) \qquad \deg R<\deg Q$$

By polynomial division, we have

$$D(x_\iota)=\begin{cases} 0 & \text{if } 0\le r=\deg P<\deg Q=n-1\\ 1 & \text{if } r=\deg P=\deg Q=n-1\\ \sum_{1\le i\le n} x_i & \text{if } r=\deg P=\deg Q+1=n\end{cases}$$

Also, for an arbitrary κ, $1 \le \kappa \le n$, we have

$$\begin{aligned}
&\frac{x_\iota^r}{\prod_{\substack{1\le i\le n\\ i\ne\iota}}(x_\iota - x_i)} = D(x_\iota) + \sum_{\substack{1\le i\le n\\ i\ne\iota}} \frac{c_i}{x_\iota - x_i}\\
\iff\quad & \frac{x_\iota^r}{(x_\iota - x_\kappa)\prod_{\substack{1\le i\le n\\ i\ne\iota\\ i\ne\kappa}}(x_\iota - x_i)} = D(x_\iota) + \sum_{\substack{1\le i\le n\\ i\ne\iota\\ i\ne\kappa}} \frac{c_i}{x_\iota - x_i} + \frac{c_\kappa}{x_\iota - x_\kappa}\\
\iff\quad & \frac{x_\iota^r(x_\iota - x_\kappa)}{(x_\iota - x_\kappa)\prod_{\substack{1\le i\le n\\ i\ne\iota\\ i\ne\kappa}}(x_\iota - x_i)} = (x_\iota - x_\kappa)D(x_\iota) + (x_\iota - x_\kappa)\sum_{\substack{1\le i\le n\\ i\ne\iota\\ i\ne\kappa}} \frac{c_i}{x_\iota - x_i} + \frac{c_\kappa(x_\iota - x_\kappa)}{x_\iota - x_\kappa}\\
\iff\quad & \frac{x_\iota^r}{\prod_{\substack{1\le i\le n\\ i\ne\iota\\ i\ne\kappa}}(x_\iota - x_i)} = (x_\iota - x_\kappa)D(x_\iota) + (x_\iota - x_\kappa)\sum_{\substack{1\le i\le n\\ i\ne\iota\\ i\ne\kappa}} \frac{c_i}{x_\iota - x_i} + c_\kappa\\
\iff\quad & c_\kappa = \frac{x_\iota^r}{\prod_{\substack{1\le i\le n\\ i\ne\iota\\ i\ne\kappa}}(x_\iota - x_i)} + (x_\kappa - x_\iota)D(x_\iota) + (x_\kappa - x_\iota)\sum_{\substack{1\le i\le n\\ i\ne\iota\\ i\ne\kappa}} \frac{c_i}{x_\iota - x_i}
\end{aligned}$$

Letting $\iota = \kappa$, we find

$$\begin{aligned}
c_\kappa &= \frac{x_\kappa^r}{\prod_{\substack{1\le i\le n\\ i\ne\kappa}}(x_\kappa - x_i)} + (x_\kappa - x_\kappa)D(x_\kappa) + (x_\kappa - x_\kappa)\sum_{\substack{1\le i\le n\\ i\ne\kappa}} \frac{c_i}{x_\kappa - x_i}\\
&= \frac{x_\kappa^r}{\prod_{\substack{1\le i\le n\\ i\ne\kappa}}(x_\kappa - x_i)}
\end{aligned}$$

And so

$$\begin{aligned}
\frac{x_\iota^r}{\prod_{\substack{1\le i\le n\\ i\ne\iota}}(x_\iota - x_i)} &= D(x_\iota) + \sum_{\substack{1\le i\le n\\ i\ne\iota}} \frac{c_i}{x_\iota - x_i}\\
&= D(x_\iota) + \sum_{\substack{1\le i\le n\\ i\ne\iota}} \frac{x_i^r}{(x_\iota - x_i)\prod_{\substack{1\le j\le n\\ j\ne i}}(x_i - x_j)}\\
&= D(x_\iota) - \sum_{\substack{1\le i\le n\\ i\ne\iota}} \frac{x_i^r}{(x_i - x_\iota)\prod_{\substack{1\le j\le n\\ j\ne i}}(x_i - x_j)}
\end{aligned}$$

or equivalently

$$D(x_\iota) = \frac{x_\iota^r}{\prod_{\substack{1\le i\le n\\ i\ne\iota}}(x_\iota - x_i)} + \sum_{\substack{1\le i\le n\\ i\ne\iota}} \frac{x_i^r}{(x_i - x_\iota)\prod_{\substack{1\le j\le n\\ j\ne i}}(x_i - x_j)}$$

letting $\iota = n$ yields

$$D(x_n)$$

$$= \frac{x_n^r}{\prod_{\substack{1\le i\le n\\ i\ne n}}(x_n - x_i)} + \sum_{\substack{1\le i\le n\\ i\ne n}} \frac{x_i^r}{(x_i - x_n)\prod_{\substack{1\le j\le n\\ j\ne i}}(x_i - x_j)}$$

$$= \sum_{1\le i\le n-1} \frac{x_i^r}{(x_i - x_n)\prod_{\substack{1\le j\le n-1\\ j\ne i}}(x_i - x_j)} + \frac{x_n^r}{\prod_{1\le i\le n-1}(x_n - x_i)}$$

$$= \frac{\sum_{1\le i\le n-1}\left(\left(\prod_{\substack{1\le j\le n-1\\ j\ne i}}\prod_{\substack{1\le k\le n-1\\ k\ne j}}(x_j - x_k)\right)\frac{x_i^r}{x_i - x_n}\right)}{\left(\prod_{1\le i\le n-1}\prod_{\substack{1\le j\le n-1\\ j\ne i}}(x_i - x_j)\right)} + \frac{x_n^r}{\prod_{1\le i\le n-1}(x_n - x_i)}$$

$$= \frac{\sum_{1\le i\le n-1}\left(\frac{1}{x_i - x_n}\left(\prod_{\substack{1\le j\le n-1\\ j\ne i}}\prod_{\substack{1\le k\le n-1\\ k\ne j}}(x_j - x_k)\right)x_i^r\right)}{\left(\prod_{1\le i\le n-1}\prod_{\substack{1\le j\le n-1\\ j\ne i}}(x_i - x_j)\right)} + \frac{x_n^r}{\prod_{1\le i\le n-1}(x_n - x_i)}$$

$$= \frac{\sum_{1\le i\le n-1}\left(\left(\prod_{\substack{1\le j\le n-1\\ j\ne i}}(x_j - x_n)\right)\left(\prod_{\substack{1\le j\le n-1\\ j\ne i}}\prod_{\substack{1\le k\le n-1\\ k\ne j}}(x_j - x_k)\right)x_i^r\right)}{\left(\prod_{1\le i\le n-1}(x_i - x_n)\right)\left(\prod_{1\le i\le n-1}\prod_{\substack{1\le j\le n-1\\ j\ne i}}(x_i - x_j)\right)} + \frac{x_n^r}{\prod_{1\le i\le n-1}(x_n - x_i)}$$

$$= \frac{\sum_{1\le i\le n-1}\left(\left(\prod_{1\le j\le n-1}(x_n - x_j)\right)\left(\prod_{\substack{1\le j\le n-1\\ j\ne i}}(x_j - x_n)\right)\left(\prod_{\substack{1\le j\le n-1\\ j\ne i}}\prod_{\substack{1\le k\le n-1\\ k\ne j}}(x_j - x_k)\right)x_i^r\right)}{\left(\prod_{1\le j\le n-1}(x_n - x_j)\right)\left(\prod_{1\le i\le n-1}(x_i - x_n)\right)\left(\prod_{1\le i\le n-1}\prod_{\substack{1\le j\le n-1\\ j\ne i}}(x_i - x_j)\right)}$$

$$+ \frac{\left(\prod_{1\le i\le n-1}(x_i - x_n)\right)\left(\prod_{1\le i\le n-1}\prod_{\substack{1\le j\le n-1\\ j\ne i}}(x_i - x_j)\right)x_n^r}{\left(\prod_{1\le j\le n-1}(x_n - x_j)\right)\left(\prod_{1\le i\le n-1}(x_i - x_n)\right)\left(\prod_{1\le i\le n-1}\prod_{\substack{1\le j\le n-1\\ j\ne i}}(x_i - x_j)\right)}$$

$$= \frac{U}{V}$$

where

$$\begin{aligned}
U &= \sum_{1\le i\le n-1}\left(\left(\prod_{1\le j\le n-1}(x_n-x_j)\right)\left(\prod_{\substack{1\le j\le n-1\\ j\ne i}}(x_j-x_n)\right)\left(\prod_{\substack{1\le j\le n-1\\ j\ne i}}\prod_{\substack{1\le k\le n-1\\ k\ne j}}(x_j-x_k)\right)x_i^r\right)\\
&\quad+\left(\left(\prod_{1\le i\le n-1}(x_i-x_n)\right)\left(\prod_{1\le i\le n-1}\prod_{\substack{1\le j\le n-1\\ j\ne i}}(x_i-x_j)\right)\right)x_n^r\\
&= \sum_{1\le i\le n-1}\left(\left(\prod_{1\le j\le n-1}(x_n-x_j)\right)\left(\prod_{\substack{1\le j\le n-1\\ j\ne i}}(x_j-x_n)\right)\left(\prod_{\substack{1\le j\le n-1\\ j\ne i}}\prod_{\substack{1\le k\le n-1\\ k\ne j}}(x_j-x_k)\right)x_i^r\right)\\
&\quad+\left(\prod_{1\le i\le n-1}\left(\left(\prod_{\substack{1\le j\le n-1\\ j\ne i}}(x_i-x_j)\right)(x_i-x_n)\right)\right)x_n^r\\
&= \sum_{1\le i\le n-1}\left(\left(\prod_{1\le j\le n-1}(x_n-x_j)\right)\left(\prod_{\substack{1\le j\le n-1\\ j\ne i}}(x_j-x_n)\right)\left(\prod_{\substack{1\le j\le n-1\\ j\ne i}}\prod_{\substack{1\le k\le n-1\\ k\ne j}}(x_j-x_k)\right)x_i^r\right)\\
&\quad+\left(\prod_{1\le i\le n-1}\prod_{\substack{1\le j\le n\\ j\ne i}}(x_i-x_j)\right)x_n^r\\
&= \sum_{1\le i\le n-1}\left(\left(\prod_{\substack{1\le j\le n-1\\ j\ne i}}\left(\left(\prod_{\substack{1\le k\le n-1\\ k\ne j}}(x_j-x_k)\right)(x_j-x_n)\right)\right)\left(\prod_{\substack{1\le k\le n\\ k\ne n}}(x_n-x_k)\right)x_i^r\right)\\
&\quad+\left(\prod_{\substack{1\le j\le n\\ j\ne n}}\prod_{\substack{1\le k\le n\\ k\ne j}}(x_j-x_k)\right)x_n^r\\
&= \sum_{1\le i\le n-1}\left(\left(\prod_{\substack{1\le j\le n-1\\ j\ne i}}\prod_{\substack{1\le k\le n\\ k\ne j}}(x_j-x_k)\right)\left(\prod_{\substack{1\le k\le n\\ k\ne n}}(x_n-x_k)\right)x_i^r\right)\\
&\quad+\left(\prod_{\substack{1\le j\le n\\ j\ne n}}\prod_{\substack{1\le k\le n\\ k\ne j}}(x_j-x_k)\right)x_n^r\\
&= \sum_{1\le i\le n-1}\left(\left(\prod_{\substack{1\le j\le n\\ j\ne i}}\prod_{\substack{1\le k\le n\\ k\ne j}}(x_j-x_k)\right)x_i^r\right)\\
&\quad+\left(\prod_{\substack{1\le j\le n\\ j\ne n}}\prod_{\substack{1\le k\le n\\ k\ne j}}(x_j-x_k)\right)x_n^r\\
&= \sum_{1\le i\le n}\left(\left(\prod_{\substack{1\le j\le n\\ j\ne i}}\prod_{\substack{1\le k\le n\\ k\ne j}}(x_j-x_k)\right)x_i^r\right)
\end{aligned}$$

and where

$$V = \left(\prod_{1\le j\le n-1}(x_n - x_j)\right)\left(\prod_{1\le i\le n-1}(x_i - x_n)\right)\left(\prod_{1\le i\le n-1}\prod_{\substack{1\le j\le n-1\\ j\ne i}}(x_i - x_j)\right)$$
$$= \left(\prod_{1\le i\le n-1}\left(\left(\prod_{\substack{1\le j\le n-1\\ j\ne i}}(x_i - x_j)\right)(x_i - x_n)\right)\right)\left(\prod_{1\le j\le n-1}(x_n - x_j)\right)$$
$$= \left(\prod_{1\le i\le n-1}\prod_{\substack{1\le j\le n\\ j\ne i}}(x_i - x_j)\right)\left(\prod_{1\le j\le n-1}(x_n - x_j)\right)$$
$$= \prod_{1\le i\le n}\prod^{\substack{1\le j\le n\\ j\ne i}}(x_i - x_j)$$

so that

$$D(x_n) = \frac{U}{V} = \frac{\sum_{1\le i\le n}\left(\left(\prod_{\substack{1\le j\le n\\ j\ne i}}\prod_{\substack{1\le k\le n\\ k\ne j}}(x_j - x_k)\right)x_i^r\right)}{\prod_{1\le i\le n}\prod_{\substack{1\le j\le n\\ j\ne i}}(x_i - x_j)} = \sum_{1\le i\le n}\frac{x_i^r}{\prod_{\substack{1\le j\le n\\ j\ne i}}(x_i - x_j)}.$$

That is,

$$\sum_{1\le i\le n}\frac{x_i^r}{\prod_{\substack{1\le j\le n\\ j\ne i}}(x_i - x_j)} = \begin{cases} 0 & \text{if } 0\le r< n-1\\ 1 & \text{if } r = n-1\\ \sum_{1\le i\le n} x_i & \text{if } r = n\end{cases}$$

as we needed to show. □

Notes: Dr. Matrix was anticipated in this discovery by L. Euler, who wrote to Christian Golbach about it on 9 November 1762. See Euler's *Institutionum Calculi Integralis* **2** (1769), §1169; and E. Waring, *Phil. Trans.* **69** (1779), 64–67... [J. J. Sylvester, *Quart. J. Math.* **1** (1857), 141–152.]

34. [*M25*] Prove that

$$\sum_{k=1}^{n}\frac{\prod_{1\le r\le n,\ r\ne m}(x+k-r)}{\prod_{1\le r\le n,\ r\ne k}(k-r)} = 1,$$

provided that $1\le m\le n$ and x is arbitrary. For example, if $n = 4$ and $m = 2$, then

$$\frac{x(x-2)(x-3)}{(-1)(-2)(-3)} + \frac{(x+1)(x-1)(x-2)}{(1)(-1)(-2)} + \frac{(x+2)x(x-1)}{(2)(1)(-1)} + \frac{(x+3)(x+1)x}{(3)(2)(1)} = 1.$$

Proposition. $\sum_{1\le i\le n}\frac{\prod_{1\le j\le n,\ j\ne k}(x+i-j)}{\prod_{1\le j\le n,\ j\ne i}(i-j)} = 1$ *provided that* $1\le k\le n$ *and* x *is arbitrary.*

Proof. We may prove

$$\sum_{1\le i\le n}\frac{\prod_{1\le j\le n,\ j\ne k}(x+i-j)}{\prod_{1\le j\le n,\ j\ne i}(i-j)} = 1$$

provided that $1\le k\le n$ and x is arbitrary, but first we shall first prove the more general result

$$\sum_{1\le i\le n}\frac{\prod_{1\le j\le n, j\ne k}(y_i - z_j)}{\prod_{1\le j\le n, j\ne i}(y_i - y_j)} = 1.$$

Let $P(y)$ be the polynomial representation of $\prod_{1\le j\le n, j\ne k}(y-z_j)$ where

$$P(y) = \prod_{\substack{1\le j\le n\\ j\ne k}} (y-z_j) = \sum_{0\le j\le n-1} c_j y^j$$

for arbitrary coefficients $c_0, c_1, \ldots, c_{n-1}$. Note that since $P(y) = y^{n-1} + \ldots$, $c_{n-1} = 1$. Then, from exercise 33, we find that

$$\begin{aligned}
\sum_{1\le i\le n} \frac{\prod_{1\le j\le n, j\ne k}(y_i - z_j)}{\prod_{1\le j\le n, j\ne i}(y_i - y_j)} &= \sum_{1\le i\le n} \frac{P(y_i)}{\prod_{1\le j\le n, j\ne i}(y_i - y_j)} \\
&= \sum_{1\le i\le n} \frac{\sum_{0\le j\le n-1} c_j y_i^j}{\prod_{1\le j\le n, j\ne i}(y_i - y_j)} \\
&= \sum_{1\le j\le n} \frac{\sum_{0\le i\le n-1} c_i y_j^i}{\prod_{1\le k\le n, k\ne j}(y_j - y_k)} \\
&= \sum_{0\le i\le n-1} c_i \sum_{1\le j\le n} \frac{y_j^i}{\prod_{1\le k\le n, k\ne j}(y_j - y_k)} \\
&= \left(\sum_{0\le i< n-1} c_i \left(\sum_{1\le j\le n} \frac{y_j^i}{\prod_{1\le k\le n, k\ne j}(y_j - y_k)} \right) \right) \\
&\quad + c_{n-1} \left(\sum_{1\le j\le n} \frac{y_j^{n-1}}{\prod_{1\le k\le n, k\ne j}(y_j - y_k)} \right) \\
&= \left(\sum_{0\le i< n-1} c_i\,(0) \right) + c_{n-1}\,(1) \\
&= c_{n-1} \\
&= 1.
\end{aligned}$$

Letting $y_i = i$ and $z_i = i + x$ for $1 \le i \le n$, $1 \le k \le n$, and x arbitrary, we have that

$$\sum_{1\le i\le n} \frac{\prod_{1\le j\le n,\ j\ne k}(x+i-j)}{\prod_{1\le j\le n,\ j\ne i}(i-j)} = 1$$

as we needed to show. □

35. [*HM20*] The notation $\sup_{R(j)} a_j$ is used to denote the least upper bound of the elements a_j, in a manner analogous to the $\sum$- and $\prod$-notations. (When $R(j)$ is satisfied for only finitely many j, the notation $\max_{R(j)} a_j$ is often used to denote the same quantity.) Show how rules (a), (b), (c), and (d) can be adapted for manipulation of *this* notation. In particular discuss the following analog of rule (a):

$$(\sup_{R(j)} a_i) + (\sup_{S(j)} b_j) = \sup_{R(i)}(\sup_{S(j)}(a_i + b_j)),$$

and give a suitable definition for the notation when $R(j)$ is satisfied for *no* j.

We have the following analogs for the least upper bound: an *additive law*:

$$(\sup_{R(i)} a_i) + (\sup_{S(j)} b_j) = \sup_{R(i)}(\sup_{S(j)}(a_i + b_j))$$

as well as a *multiplicative law*:

$$(\sup_{R(i)} a_i)(\sup_{S(j)} b_j) = \sup_{R(i)}(\sup_{S(j)}(a_i b_j))$$

provided a_i and a_j are nonnegative; *change of variable*:

$$\sup_{R(i)} a_i = \sup_{R(j)} = \sup_{R(p(j))} a_{p(j)};$$

interchanging order of bound:

$$\sup_{R(i)} \sup_{S(j)} a_{ij} = \sup_{S(j)} \sup_{R(i)} a_{ij};$$

and *manipulating the domain*:

$$\sup(\sup_{R(j)} a_j, \sup_{S(j)} a_j) = \sup_{R(j)\vee S(j)} a_j.$$

A suitable definition for the notation when $R(j)$ is satisfied for *no* j would be

$$\sup_{j\in\emptyset} = -\infty$$

since $-\infty$ acts as the identity element for the least upper bound, in that $\sup(\sup_{j\in\emptyset} a_j, \sup_{R(j)} a_j) = \sup_{R(j)} a_j$.

36. [*M23*] Show that the determinant of the combinatorial matrix is $x^{n-1}(x + ny)$.

Proposition. $\det[y + \delta_{ij}x]_n = x^{n-1}(ny + x)$.

Proof. For

$$\mathbf{A}_n = [a_{ij}]_n = [y + \delta_{ij}x]_n = \begin{bmatrix} y+x & y & \cdots & y \\ y & y+x & \cdots & y \\ \vdots & \vdots & \ddots & \vdots \\ y & y & \cdots & y+x \end{bmatrix}_n ,$$

we must show that

$$\det[a_{ij}]_n = x^{n-1}(ny + x).$$

Let

$$a'_{ij} = \begin{cases} a_{i1} & \text{if } j = 1 \\ a_{ij} - a_{i1} & \text{if } 2 \le j \le n \end{cases}$$

and

$$a''_{ij} = \begin{cases} a'_{1j} + \sum_{2\le k\le n} a'_{kj} & \text{if } i = 1 \\ a'_{ij} & \text{if } 2 \le i \le n \end{cases}$$

so that $\det[a_{ij}]_n = \det[a''_{ij}]_n$ where

$$a''_{ij} = \begin{cases} ny + x & \text{if } i = 1,\ j = 1 \\ 0 & \text{if } i = 1,\ 2 \le j \le n \\ y & \text{if } 2 \le i \le n,\ j = 1 \\ \delta_{ij}x & \text{if } 2 \le i \le n,\ 2 \le j \le n \end{cases}$$

since:

$$\begin{aligned}
a'_{11} &= a_{11} = y + x && i = 1, j = 1 \\
a'_{1j} &= -x && i = 1, 2 \le j \le n \\
a'_{i1} &= a_{i1} = y && 2 \le i \le n, j = 1 \\
a'_{ij} &= \delta_{ij}x && 2 \le i \le n, 2 \le j \le n \\
a''_{11} &= a'_{11} + \sum_{2\le k\le n} a'_{k1} = y + x + \sum_{2\le k\le n} y = ny + x && i = 1, j = 1 \\
a''_{1j} &= a'_{1j} + \sum_{2\le k\le n} a'_{kj} = -x + \sum_{2\le k\le n} \delta_{kj}x = 0 && i = 1, 2 \le j \le n \\
a''_{i1} &= a'_{i1} = y && 2 \le i \le n, j = 1 \\
a''_{ij} &= a'_{ij} = \delta_{ij}x && 2 \le i \le n, 2 \le j \le n
\end{aligned}$$

Then, since $[a''_{ij}]_n$ is a triangular matrix ($a''_{ij} = 0$ whenever $i < j$), we have that

$$\begin{aligned}
\det[a_{ij}]_n &= \det[a''_{ij}]_n \\
&= \prod_{1\le k\le n} a''_{kk} \\
&= a''_{11} \prod_{2\le k\le n} a''_{kk} \\
&= (ny+x) \prod_{2\le k\le n} \delta_{kk} x \\
&= (ny+x) \prod_{2\le k\le n} x \\
&= (ny+x)x^{n-1} \\
&= x^{n-1}(ny+x)
\end{aligned}$$

as we needed to show. □

▶ **37.** [*M24*] Show that the determinant of Vandermonde's matrix is

$$\prod_{1\le j\le n} x_j \prod_{1\le i<j\le n} (x_j - x_i).$$

Proposition. $\det[x_j^i]_n = \prod_{1\le i\le n} x_i \prod_{1\le j<i\le n}(x_i - x_j)$.

Proof. For

$$\mathbf{A}_n = [a_{ij}]_n = [x_j^i]_n = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^n & x_2^n & \cdots & x_n^n \end{bmatrix}_n,$$

we must show that

$$\det[a_{ij}]_n = \prod_{1\le i\le n} x_i \prod_{1\le j<i\le n} (x_i - x_j).$$

Let

$$a'_{ij} = \begin{cases} a_{i1} & \text{if } j = 1 \\ a_{ij} - a_{i1} & \text{if } 2 \le j \le n \end{cases}$$

and

$$a''_{ij} = \begin{cases} a'_{11} & \text{if } i = 1, j = 1 \\ a'_{1j} & \text{if } i = 1, 2 \le j \le n \\ a'_{i1} - x_1 a'_{(i-1)1} & \text{if } 2 \le i \le n, j = 1, \\ a'_{ij} - x_1 a'_{(i-1)j} & \text{if } 2 \le i \le n, 2 \le j \le n, \end{cases}$$

so that $\det[a_{ij}] = \det[a''_{ij}]$ where

$$a''_{ij} = \begin{cases} x_1 & \text{if } i = 1,\ j = 1 \\ x_j - x_1 & \text{if } i = 1,\ 2 \le j \le n \\ 0 & \text{if } 2 \le i \le n,\ j = 1 \\ x_j^{i-1}(x_j - x_1) & \text{if } 2 \le i \le n,\ 2 \le j \le n \end{cases}$$

since:

$a'_{11} = a_{11} = x_1$	$i = 1, j = 1$
$a'_{1j} = a_{1j} - a_{11} = x_j - x_1$	$i = 1, 2 \leq j \leq n$
$a'_{i1} = a_{i1} = x_1^i$	$2 \leq i \leq n, j = 1$
$a'_{ij} = a_{ij} - a_{i1} = x_j^i - x_1^i$	$2 \leq i \leq n, 2 \leq j \leq n$
$a''_{11} = a'_{11} = a_{11} = x_1$	$i = 1, j = 1$
$a''_{1j} = a'_{1j} = a_{1j} - a_{11} = x_j - x_1$	$i = 1, 2 \leq j \leq n$
$a''_{i1} = a'_{i1} - x_1 a'_{(i-1)1} = a_{i1} - x_1 a_{(i-1)1}) = 0$	$2 \leq i \leq n, j = 1$
$a''_{ij} = a'_{ij} - x_1 a'_{(i-1)j} = a_{ij} - a_{i1} - x_1(a_{(i-1)j} - a_{(i-1)1}) = x_j^{i-1}(x_j - x_1)$	$2 \leq i \leq n, 2 \leq j \leq n$

Let

$$q_{ij} = \begin{cases} x_{j+1} - x_1 & \text{if } i = j, 1 \leq i, j \leq n-1 \\ 0 & \text{otherwise} \end{cases}$$

so that $\operatorname{minor}([a'']_n, 1, 1) = ([q_{ij}]_{n-1} \operatorname{minor}([a]_n, 1, 1)^T)^T$ and $\det[q_{ij}]_{n-1} = \prod_{1 \leq k \leq n-1}(x_{k+1} - x_1)$. Then

$$\begin{aligned}
\det[a_{ij}]_n &= \det[a''_{ij}]_n \\
&= \sum_{1 \leq i \leq n} a''_{i1} \operatorname{cofactor}(a''_{i1}) \\
&= a''_{11} \operatorname{cofactor}(a''_{11}) + \sum_{2 \leq i \leq n} a''_{i1} \operatorname{cofactor}(a''_{i1}) \\
&= x_1(-1)^{1+1} \det \operatorname{minor}([a'']_n, 1, 1) + 0 \\
&= x_1 \det \operatorname{minor}([a'']_n, 1, 1) \\
&= x_1 \det(([q_{ij}]_{n-1} \operatorname{minor}([a]_n, 1, 1)^T)^T) \\
&= x_1 \det([q_{ij}]_{n-1} \operatorname{minor}([a]_n, 1, 1)^T) \\
&= x_1 \det[q_{ij}]_{n-1} \det(\operatorname{minor}([a]_n, 1, 1)^T) \\
&= x_1 \det[q_{ij}]_{n-1} \det \operatorname{minor}([a]_n, 1, 1) \\
&= x_1 \left(\prod_{1 \leq k \leq n-1} (x_{k+1} - x_1) \right) \det \operatorname{minor}([a]_n, 1, 1).
\end{aligned}$$

We shall finally use this recursive identity to give a proof by mathematical induction on n.

If $n = 1$, then clearly

$$\det[a_{ij}]_1 = \det[a_{11}]_1 = x_1 = \prod_{1 \leq i \leq n} x_i \prod_{1 \leq j < i \leq 1} (x_i - x_j).$$

Then, assuming that

$$\det[a_{ij}]_k = \prod_{1 \leq i \leq k} x_i \prod_{1 \leq j < i \leq k} (x_i - x_j)$$

or equivalently that

$$\det \operatorname{minor}([a]_{k+1}, 1, 1) = \prod_{2 \leq i \leq k+1} x_i \prod_{2 \leq j < i \leq k+1} (x_i - x_j)$$

we must show that

$$\det[a_{ij}]_{k+1} = \prod_{1\le i\le k+1} x_i \prod_{1\le j<i\le k+1} (x_i - x_j).$$

But

$$\begin{aligned}
\det[a_{ij}]_{k+1} &= x_1 \left(\prod_{1\le i\le k} (x_{i+1} - x_1) \right) \det[a_{(i+1)(j+1)}]_k \\
&= x_1 \left(\prod_{1\le i\le k} (x_{i+1} - x_1) \right) \left(\prod_{2\le i\le k+1} x_i \prod_{2\le j<i\le k+1} (x_i - x_j) \right) \\
&= \left(x_1 \prod_{2\le i\le k+1} (x_i - x_1) \right) \left(\prod_{2\le i\le k+1} x_i \prod_{2\le j<i\le k+1} (x_i - x_j) \right) \\
&= \left(\prod_{1\le i\le 1} x_i \prod_{1\le j<i\le k+1} (x_i - x_j) \right) \left(\prod_{2\le i\le k+1} x_i \prod_{2\le j<i\le k+1} (x_i - x_j) \right) \\
&= \prod_{1\le i\le k+1} x_i \prod_{1\le j<i\le k+1} (x_i - x_j)
\end{aligned}$$

as we needed to show. □

▶ **38.** [*M25*] Show that the determinant of Cauchy's matrix is

$$\prod_{1\le i<j\le n} (x_j - x_i)(y_j - y_i) \Big/ \prod_{1\le i,j\le n} (x_i + y_i).$$

Proposition. $\det[1/(x_i + y_j)]_n = \prod_{1\le i<j\le n}(x_j - x_i)(y_j - y_i) \Big/ \prod_{1\le i,j\le n}(x_i + y_j)$.

Proof. For

$$\mathbf{A}_n = [a_{ij}]_n = [1/(x_i + y_i)]_n = \begin{bmatrix} 1/(x_1+y_1) & 1/(x_1+y_2) & \cdots & 1/(x_1+y_n) \\ 1/(x_2+y_1) & 1/(x_2+y_2) & \cdots & 1/(x_2+y_n) \\ \vdots & \vdots & \ddots & \vdots \\ 1/(x_n+y_1) & 1/(x_n+y_2) & \cdots & 1/(x_n+y_n) \end{bmatrix}_n,$$

we must show that

$$\det[a_{ij}]_n = \prod_{1\le i<j\le n} (x_j - x_i)(y_j - y_i) \Big/ \prod_{1\le i,j\le n} (x_i + y_j).$$

Let

$$a'_{ij} = \begin{cases} a_{i1} & \text{if } j = 1 \\ a_{ij} - a_{i1} & \text{if } 2 \le j \le n \end{cases}$$

so that $\det[a_{ij}] = \det[a'_{ij}]$ where

$$a'_{ij} = \begin{cases} 1/(x_1+y_1) & \text{if } i = 1,\ j = 1 \\ ((y_1 - y_j)/(x_1+y_1))(1/(x_1+y_j)) & \text{if } i = 1,\ 2 \le j \le n \\ 1/(x_1+y_1) & \text{if } 2 \le i \le n,\ j = 1 \\ ((y_1 - y_j)/(x_i+y_1))(1/(x_i+y_j)) & \text{if } 2 \le i \le n,\ 2 \le j \le n \end{cases}$$

since:

$$\begin{array}{ll} a'_{11} = a_{11} = 1/(x_1 + y_1) & i = 1, j = 1 \\ a'_{1j} = a_{1j} - a_{11} = ((y_1 - y_j)/(x_1 + y_1))(1/(x_1 + y_j)) & i = 1, 2 \le j \le n \\ a'_{i1} = a_{i1} = 1/(x_i + y_1) & 2 \le i \le n, j = 1 \\ a'_{ij} = a_{ij} - a_{i1} = ((y_1 - y_j)/(x_i + y_1))(1/(x_i + y_j)) & 2 \le i \le n, 2 \le j \le n \end{array}$$

Let

$$b_{ij} = \begin{cases} 1 & \text{if } j = 1 \\ 1/(x_i + y_j) & \text{otherwise,} \end{cases}$$

$$p_{ij} = \begin{cases} 1/(x_i + y_1) & \text{if } i = j, 1 \le i, j \le n \\ 0 & \text{otherwise,} \end{cases}$$

and

$$q_{ij} = \begin{cases} 1 & \text{if } i = j = 1 \\ y_1 - y_j & \text{if } i = j, 2 \le i, j \le n \\ 0 & \text{otherwise} \end{cases}$$

so that

$$[a'_{ij}]_n = ([q_{ij}]_n([p_{ij}]_n[b_{ij}]_n)^T)^T$$

and:

$$\det[p_{ij}]_n = \prod_{1 \le k \le n} \frac{1}{x_k + y_1}$$
$$\det[q_{ij}]_n = \prod_{2 \le k \le n} y_1 - y_k$$

Also let

$$b'_{ij} = \begin{cases} b_{1j} & \text{if } i = 1 \\ b_{ij} - b_{1j} & \text{if } 2 \le i \le n \end{cases}$$

so that $\det[b_{ij}] = \det[b'_{ij}]$ where

$$b'_{ij} = \begin{cases} 1 & \text{if } i = 1,\ j = 1 \\ 1/(x_i + y_j) & \text{if } i = 1,\ 2 \le j \le n \\ 0 & \text{if } 2 \le i \le n,\ j = 1 \\ ((x_1 - x_i)/(x_1 + y_j))(1/(x_i + y_j)) & \text{if } 2 \le i \le n,\ 2 \le j \le n \end{cases}$$

since:

$$\begin{array}{ll} b'_{11} = b_{11} = 1 & i = 1, j = 1 \\ b'_{1j} = b_{1j} = 1/(x_i + y_j) & i = 1, 2 \le j \le n \\ b'_{i1} = b_{i1} - b_{11} = 0 & 2 \le i \le n, j = 1 \\ b'_{ij} = b_{ij} - b_{1j} = ((x_1 - x_i)/(x_1 + y_j))(1/(x_i + y_j)) & 2 \le i \le n, 2 \le j \le n \end{array}$$

Also let

$$r_{ij} = \begin{cases} 1 & \text{if } i = j = 1 \\ x_1 - x_i & \text{if } i = j, 2 \le i, j \le n \\ 0 & \text{otherwise} \end{cases}$$

and

$$s_{ij} = \begin{cases} 1 & \text{if } i = j = 1 \\ 1/(x_1 + y_j) & \text{if } i = j, 2 \leq i, j \leq n \\ 0 & \text{otherwise} \end{cases}$$

so that

$$\operatorname{minor}([b']_n, 1, 1) = ([s_{ij}]_{n-1}([r_{ij}]_{n-1} \operatorname{minor}([a]_n, 1, 1))^T)^T$$

and:

$$\det[r_{ij}]_{n-1} = \prod_{2 \leq k \leq n} x_1 - x_k$$

$$\det[s_{ij}]_{n-1} = \prod_{2 \leq k \leq n} \frac{1}{x_1 + y_j}$$

Then

$$\begin{aligned}
\det[a_{ij}] &= \det[a''_{ij}]_n \\
&= \det(([q_{ij}]_n([p_{ij}]_n[b_{ij}]_n)^T)^T) \\
&= \det([q_{ij}]_n)\det([p_{ij}]_n[b_{ij}]_n) \\
&= \det([q_{ij}]_n)\det([p_{ij}]_n)\det([b_{ij}]_n) \\
&= \left(\prod_{2\le k\le n} y_1 - y_k\right)\det([p_{ij}]_n)\det([b_{ij}]_n) \\
&= \left(\prod_{2\le k\le n} y_1 - y_k\right)\left(\prod_{1\le k\le n}\frac{1}{x_k+y_1}\right)\det([b_{ij}]_n) \\
&= \left(\prod_{2\le k\le n} y_1 - y_k\right)\left(\prod_{1\le k\le n}\frac{1}{x_k+y_1}\right)\det([b'_{ij}]_n) \\
&= \left(\prod_{2\le k\le n} y_1 - y_k\right)\left(\prod_{1\le k\le n}\frac{1}{x_k+y_1}\right)\left(b'_{11}\operatorname{cofactor}(b'_{11}) + \sum_{2\le i\le n} b'_{i1}\operatorname{cofactor}(b'_{i1})\right) \\
&= \left(\prod_{2\le k\le n} y_1 - y_k\right)\left(\prod_{1\le k\le n}\frac{1}{x_k+y_1}\right)(b'_{11}\operatorname{cofactor}(b'_{11}) + 0) \\
&= \left(\prod_{2\le k\le n} y_1 - y_k\right)\left(\prod_{1\le k\le n}\frac{1}{x_k+y_1}\right)(\det(\operatorname{minor}([b']_n,1,1)) + 0) \\
&= \left(\prod_{2\le k\le n} y_1 - y_k\right)\left(\prod_{1\le k\le n}\frac{1}{x_k+y_1}\right)\det(([s_{ij}]_{n-1}([r_{ij}]_{n-1}\operatorname{minor}([a]_n,1,1))^T)^T) \\
&= \left(\prod_{2\le k\le n} y_1 - y_k\right)\left(\prod_{1\le k\le n}\frac{1}{x_k+y_1}\right)\det([s_{ij}]_{n-1})\det([r_{ij}]_{n-1}\operatorname{minor}([a]_n,1,1)) \\
&= \left(\prod_{2\le k\le n} y_1 - y_k\right)\left(\prod_{1\le k\le n}\frac{1}{x_k+y_1}\right)\det([s_{ij}]_{n-1})\det([r_{ij}]_{n-1})\det(\operatorname{minor}([a]_n,1,1)) \\
&= \left(\prod_{2\le k\le n} y_1 - y_k\right)\left(\prod_{1\le k\le n}\frac{1}{x_k+y_1}\right)\left(\prod_{2\le k\le n}\frac{1}{x_1+y_j}\right)\det([r_{ij}]_{n-1})\det(\operatorname{minor}([a]_n,1,1)) \\
&= \left(\prod_{2\le k\le n} y_1 - y_k\right)\left(\prod_{1\le k\le n}\frac{1}{x_k+y_1}\right)\left(\prod_{2\le k\le n}\frac{1}{x_1+y_j}\right)\left(\prod_{2\le k\le n} x_1 - x_k\right)\det(\operatorname{minor}([a]_n,1,1)) \\
&= \frac{\prod_{2\le i\le n}(x_i-x_1)(y_i-y_1)}{\prod_{1\le i,j\le n}(x_i+y_1)(x_1+y_j)}\det(\operatorname{minor}([a]_n,1,1)).
\end{aligned}$$

We shall finally use this recursive identity to give a proof by mathematical induction on n.

If $n = 1$, then clearly

$$\det[a_{ij}]_1 = \det[a_{11}]_1 = 1/(x_1+y_1) = \prod_{1\le i<j\le 1}(x_j-x_i)(y_j-y_i)\Big/\prod_{1\le i,j\le 1}(x_i+y_j).$$

Then, assuming that

$$\det[a_{ij}]_k = \prod_{1\le i<j\le k}(x_j - x_i)(y_j - y_i) \Big/ \prod_{1\le i,j\le k}(x_i + y_j)$$

or equivalently that

$$\det\operatorname{minor}([a]_{k+1}, 1, 1) = \prod_{2\le i<j\le k+1}(x_j - x_i)(y_j - y_i) \Big/ \prod_{2\le i,j\le k+1}(x_i + y_j)$$

we must show that

$$\det[a_{ij}]_{k+1} = \prod_{1\le i<j\le k+1}(x_j - x_i)(y_j - y_i) \Big/ \prod_{1\le i,j\le k+1}(x_i + y_j).$$

But

$$\begin{aligned}
\det[a_{ij}]_{k+1} &= \frac{\prod_{2\le i\le k+1}(x_i - x_1)(y_i - y_1)}{\prod_{1\le i,j\le k+1}(x_i + y_1)(x_1 + y_j)} \det\operatorname{minor}([a]_{k+1}, 1, 1) \\
&= \frac{\prod_{2\le i\le k+1}(x_i - x_1)(y_i - y_1)}{\prod_{1\le i,j\le k+1}(x_i + y_1)(x_1 + y_j)} \frac{\prod_{2\le i<j\le k+1}(x_j - x_i)(y_j - y_i)}{\prod_{2\le i,j\le k+1}(x_i + y_j)} \\
&= \frac{\left(\prod_{2\le j\le k+1}(x_j - x_1)(y_j - y_1)\right)\left(\prod_{2\le i<j\le k+1}(x_j - x_i)(y_j - y_i)\right)}{\left(\prod_{1\le i,j\le k+1}(x_i + y_1)(x_1 + y_j)\right)\left(\prod_{2\le i,j\le k+1}(x_i + y_j)\right)} \\
&= \frac{\left(\prod_{1\le i<j\le k+1}(x_j - x_i)(y_j - y_i)\right)\left(\prod_{2\le i<j\le k+1}(x_j - x_i)(y_j - y_i)\right)}{\left(\prod_{1\le i,j\le k+1}(x_i + y_1)(x_1 + y_j)\right)\left(\prod_{2\le i,j\le k+1}(x_i + y_j)\right)} \\
&= \prod_{1\le i<j\le k+1}(x_j - x_i)(y_j - y_i) \Big/ \prod_{1\le i,j\le k+1}(x_i + y_j)
\end{aligned}$$

as we needed to show. □

39. [*M23*] Show that the inverse of a combinatorial matrix is a combinatorial matrix with the entries $b_{ij} = (-y + \delta_{ij}(x + ny))/x(x + ny)$.

Proposition. $[y + \delta_{ij}x]_n^{-1} = \left[\frac{-y+\delta_{ij}(x+ny)}{x(x+ny)}\right]_n$.

Proof. For

$$\mathbf{A}_n = [a_{ij}]_n = [y + \delta_{ij}x]_n = \begin{bmatrix} y+x & y & \cdots & y \\ y & y+x & \cdots & y \\ \vdots & \vdots & \ddots & \vdots \\ y & y & \cdots & y+x \end{bmatrix}_n,$$

we must show that

$$[a_{ij}]_n^{-1} = \left[\frac{-y + \delta_{ij}(x + ny)}{x(x + ny)}\right]_n.$$

Let $\mathbf{I}_n = [\delta_{ij}]_n$ and $\mathbf{J}_n = [1]_n$ so that $\mathbf{A}_n = y\mathbf{J}_n + x\mathbf{I}_n$, and note that $\mathbf{J}_n^2 = n\mathbf{J}_n$. Then

$$\begin{aligned}
\mathbf{A}_n(-y\mathbf{J}_n + x\mathbf{I}_n) &= x^2\mathbf{I}_n - ny^2\mathbf{J}_n && \iff \\
\mathbf{A}_n(-y\mathbf{J}_n + x\mathbf{I}_n) + ny^2\mathbf{J}_n &= x^2\mathbf{I}_n && \iff \\
\mathbf{A}_n(-y\mathbf{J}_n + x\mathbf{I}_n) + ny\mathbf{I}_n y\mathbf{J}_n &= x^2\mathbf{I}_n && \iff \\
\mathbf{A}_n(-y\mathbf{J}_n + x\mathbf{I}_n) + ny\mathbf{I}_n y\mathbf{J}_n + xny\mathbf{I}_n &= x^2\mathbf{I}_n + xny\mathbf{I}_n && \iff \\
\mathbf{A}_n(-y\mathbf{J}_n + x\mathbf{I}_n) + ny\mathbf{I}_n(y\mathbf{J}_n + x\mathbf{I}_n) &= x^2\mathbf{I}_n + xny\mathbf{I}_n && \iff \\
\mathbf{A}_n(-y\mathbf{J}_n + x\mathbf{I}_n) + ny\mathbf{I}_n\mathbf{A}_n &= x(x+ny)\mathbf{I}_n && \iff \\
\mathbf{A}_n(-y\mathbf{J}_n + x\mathbf{I}_n + ny\mathbf{I}_n) &= x(x+ny)\mathbf{I}_n && \iff \\
\mathbf{A}_n((x+ny)\mathbf{I}_n - y\mathbf{J}_n) &= x(x+ny)\mathbf{I}_n && \iff \\
\mathbf{A}_n\left(\frac{(x+ny)\mathbf{I}_n - y\mathbf{J}_n}{x(x+ny)}\right) &= \mathbf{I}_n.
\end{aligned}$$

That is, for

$$\begin{aligned}
\mathbf{B}_n &= \frac{(x+ny)\mathbf{I}_n - y\mathbf{J}_n}{x(x+ny)} \\
&= \frac{(x+ny)[\delta_{ij}]_n - y[1]_n}{x(x+ny)} \\
&= \left[\frac{(x+ny)\delta_{ij} - y}{x(x+ny)}\right]_n \\
&= \left[\frac{-y + \delta_{ij}(x+ny)}{x(x+ny)}\right]_n,
\end{aligned}$$

$\mathbf{A}_n\mathbf{B}_n = \mathbf{I}_n$, or equivalently, that

$$\mathbf{A}_n^{-1} = \left[\frac{-y + \delta_{ij}(x+ny)}{x(x+ny)}\right]_n$$

as we needed to show. □

40. [*M24*] Show that the inverse of Vandermonde's matrix is given by

$$b_{ij} = \Bigg(\sum_{\substack{1\le k_1<\cdots<k_{n-j}\le n\\ k_1,\ldots,k_{n-j}\ne i}} (-1)^{j-1} x_{k_1}\ldots x_{k_{n-j}}\Bigg)\Bigg/ x_i \prod_{\substack{1\le k\le n\\ k\ne i}} (x_k - x_i).$$

Don't be dismayed by the complicated sum in the numerator—it is just the coefficient of x^{j-1} in the polynomial $(x_1 - x)\ldots(x_n - x)/(x_i - x)$.

Proposition. $[x_j^i]_n^{-1} = \left[\left(\sum_{\substack{1\le k_1<\cdots<k_{n-j}\le n\\ k_1,\ldots,k_{n-j}\ne i}} (-1)^{j-1} x_{k_1}\ldots x_{k_{n-j}}\right)\Big/ x_i \prod_{\substack{1\le k\le n\\ k\ne i}}(x_k - x_i)\right].$

Proof. Let

$$\mathbf{A}_n = [a_{ij}]_n = [x_j^i]_n = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^n & x_2^n & \cdots & x_n^n \end{bmatrix}_n.$$

We must show that

$$[a_{ij}]_n^{-1} = \left[\sum_{\substack{1\le k_1<\ldots<k_{n-j}\le n\\ k_1,\ldots,k_{n-j}\ne i}} (-1)^{j-1} x_{k_1}\cdots x_{k_{n-j}} \Bigg/ x_i \prod_{\substack{1\le k\le n\\ k\ne i}} (x_k - x_i)\right]_n.$$

Let $[b_{ij}]_n = [a_{ij}]_n^{-1}$, so that by the definition of inverse and matrix multiplication, we have

$$[b_{ij}]_n[a_{ij}]_n = \left[\sum_{1\le k\le n} b_{ik}a_{kj}\right]_n = \left[\sum_{1\le k\le n} b_{ik}x_j^k\right]_n = [\delta_{ij}]_n.$$

We require

$$\sum_{1\le k\le n} b_{ik}x_j^k = \delta_{ij},$$

or equivalently, given a polynomial $P_i(x) = \sum_{1\le k\le n} b_{ik}x^k$ for arbitrary x, we require

$$P_i(x) = \delta_{ij}$$

with given data points $(\delta_{i1}, x_1), (\delta_{i2}, x_2), \ldots, (\delta_{in}, x_n)$. By polynomial interpolation, expanding the polynomial to an initial but trivial $(n+1)$th variable $x_0 = 0$, with $b_{i0} = b_{0j} = 0$, in order to obtain a complete set of differences, we have

$$\begin{aligned}
P_i(x) &= \sum_{0\le k\le n}\left(\delta_{ik}\prod_{\substack{0\le m\le n\\ m\ne k}}\frac{x-x_m}{x_k-x_m}\right)\\
&= \sum_{\substack{0\le k\le n\\ k=i}}\left(\delta_{ik}\prod_{\substack{0\le m\le n\\ m\ne k}}\frac{x-x_m}{x_k-x_m}\right) + \sum_{\substack{0\le k\le n\\ k\ne i}}\left(\delta_{ik}\prod_{\substack{0\le m\le n\\ m\ne k}}\frac{x-x_m}{x_k-x_m}\right)\\
&= \delta_{ii}\prod_{\substack{0\le m\le n\\ m\ne i}}\frac{x-x_m}{x_i-x_m} + 0\\
&= \prod_{\substack{0\le k\le n\\ k\ne i}}\frac{x-x_k}{x_i-x_k}\\
&= \frac{x-x_0}{x_i-x_0}\prod_{\substack{1\le k\le n\\ k\ne i}}\frac{x-x_k}{x_i-x_k}\\
&= \frac{x}{x_i}\prod_{\substack{1\le k\le n\\ k\ne i}}\frac{x_k-x}{x_k-x_i}.
\end{aligned}$$

For $x = x_j$, we have

$$\sum_{1\le k\le n} b_{ik}x_j^k = P_i(x_j) = \frac{x_j}{x_i}\prod_{\substack{1\le k\le n\\ k\ne i}}\frac{x_k-x_j}{x_k-x_i} = \delta_{ij}.$$

What is left is to find the coefficients b_{ij}. By de Moivre's work[1], we may do so.

The real and different roots of $P_i(x)$ are exactly those x_r where $r \ne i$, since in such a case, we have

$$P_i(x_r) = \frac{x_r}{x_i}\prod_{\substack{1\le k\le n\\ k\ne i}}\frac{x_k-x_r}{x_k-x_i} = (x_r - x_r)\frac{x_r}{x_i}\prod_{\substack{1\le k\le n\\ k\ne i\\ k\ne r}}\frac{x_k-x_r}{x_k-x_i} = 0.$$

Let these roots be denoted by $x_{r_1}, x_{r_2}, \ldots, x_{r_n}$.

[1] Anders Hald, *History of Probability and Statistics and Their Applications before 1750* (John Wiley & Sons, 2003), pp. 429–430.

Since matrix multiplication with inverse is commutative, We also have

$$\sum_{1\le k\le n} b_{ik}x_j^k = \sum_{1\le k\le n} x_k^i b_{kj} = \delta_{ij}.$$

By de Moivre's identities, with our trivially expanded polynomial,

$$\delta_{ij} = \sum_{0\le k\le n} x_k^i b_{kj}$$
$$\iff$$
$$b_{kj} = \sum_{1\le m\le n} (-1)^m \sum_{\substack{1\le r_1<\cdots<r_m\le n\\ r_1,\ldots,r_m\ne i}} x_{r_1}\cdots x_{r_m}\delta_{(n-m)j} \Big/ x_k \prod_{\substack{1\le m\le n\\ m\ne k}} (x_k - x_m)$$

we then have, since $\delta_{(n-k)j}$ requires $j = n-k$ and since $n-j$ and $j-1$ have opposite parity, that

$$\begin{aligned}
b_{ij} &= \sum_{1\le k\le n} (-1)^k \sum_{\substack{1\le r_1<\cdots<r_k\le n\\ r_1,\ldots,r_k\ne i}} x_{r_1}\cdots x_{r_k}\delta_{(n-k)j} \Big/ x_i \prod_{\substack{1\le k\le n\\ k\ne i}} (x_i - x_k)\\
&= (-1)^{n-j} \sum_{\substack{1\le r_1<\cdots<r_{n-j}\le n\\ r_1,\ldots,r_{n-j}\ne i}} x_{r_1}\cdots x_{r_{n-j}} \Big/ x_i \prod_{\substack{1\le k\le n\\ k\ne i}} (x_i - x_k)\\
&= (-1)(-1)^{j-1} \sum_{\substack{1\le r_1<\cdots<r_{n-j}\le n\\ r_1,\ldots,r_{n-j}\ne i}} x_{r_1}\ldots x_{r_{n-j}} \Big/ (-1)x_i \prod_{\substack{1\le k\le n\\ k\ne i}} (x_k - x_i)\\
&= \sum_{\substack{1\le r_1<\cdots<r_{n-j}\le n\\ r_1,\ldots,r_{n-j}\ne i}} (-1)^{j-1}x_{r_1}\cdots x_{r_{n-j}} \Big/ x_i \prod_{\substack{1\le k\le n\\ k\ne i}} (x_k - x_i).
\end{aligned}$$

Therefore

$$[a_{ij}]_n^{-1} = \left[\sum_{\substack{1\le k_1<\ldots<k_{n-j}\le n\\ k_1,\ldots,k_{n-j}\ne i}} (-1)^{j-1}x_{k_1}\cdots x_{k_{n-j}} \Big/ x_i \prod_{\substack{1\le k\le n\\ k\ne i}} (x_k - x_i)\right]_n$$

as we needed to show. □

[A. de Moivre, *The Doctrine of Chances*, 2nd edition (London: 1738), 197–199.]

41. [*M26*] Show that the inverse of Cauchy's matrix is given by

$$b_{ij} = \left(\prod_{1\le k\le n}(x_j+y_k)(x_k+y_i)\right) \Big/ (x_j+y_i)\left(\prod_{\substack{1\le k\le n\\ k\ne j}}(x_j-x_k)\right)\left(\prod_{\substack{1\le k\le n\\ k\ne i}}(y_i-y_k)\right).$$

Let

$$\mathbf{A}_n = [a_{ij}]_n = [1/(x_i+y_i)]_n = \begin{bmatrix} 1/(x_1+y_1) & 1/(x_1+y_2) & \cdots & 1/(x_1+y_n)\\ 1/(x_2+y_1) & 1/(x_2+y_2) & \cdots & 1/(x_2+y_n)\\ \vdots & \vdots & \ddots & \vdots\\ 1/(x_n+y_1) & 1/(x_n+y_2) & \cdots & 1/(x_n+y_n)\end{bmatrix}_n,$$

We must show that

$$[a_{ij}]_n^{-1} = \left[\left(\prod_{1\le k\le n} (x_j + y_k)(x_k + y_i) \right) \Big/ (x_j + y_i) \left(\prod_{\substack{1\le k\le n \\ k\ne j}} (x_j - x_k) \right) \left(\prod_{\substack{1\le k\le n \\ k\ne i}} (y_i - y_k) \right) \right]_n .$$

But, by the definition of inverse,

$$
\begin{aligned}
[a_{ij}]_n^{-1} &= \frac{[\text{cofactor}(a_{ij})]_n^T}{\det[a_{ij}]_n} \\
&= \frac{[\text{cofactor}(a_{ji})]_n}{\prod_{1\le u<v\le n}(x_v - x_u)(y_v - y_u) \Big/ \prod_{1\le u,v\le n}(x_u + y_v)} \\
&= \frac{(-1)^{j+i}\det\text{minor}([a]_n, j, i)}{\prod_{1\le u<v\le n}(x_v - x_u)(y_v - y_u) \Big/ \prod_{1\le u,v\le n}(x_u + y_v)} \\
&= \frac{(-1)^{j+i}\prod_{\substack{1\le u<v\le n\\ u\ne j, u\ne i\\ v\ne j, v\ne i}}(x_v - x_u)(y_v - y_u) \Big/ \prod_{\substack{1\le u,v\le n\\ u\ne j\\ v\ne i}}(x_u + y_v)}{\prod_{1\le u<v\le n}(x_v - x_u)(y_v - y_u) \Big/ \prod_{1\le u,v\le n}(x_u + y_v)} \\
&= \frac{(-1)^{j+i}\left(\prod_{\substack{1\le u<v\le n\\ u\ne j\\ v\ne j}}(x_u - x_v)\right)\left(\prod_{\substack{1\le u<v\le n\\ u\ne i\\ v\ne i}}(y_u - y_v)\right) \Big/ \prod_{\substack{1\le u,v\le n\\ u\ne j\\ v\ne i}}(x_u + y_v)}{\left(\prod_{1\le u<v\le n}(x_u - x_v)\right)\left(\prod_{1\le u<v\le n}(y_u - y_v)\right) \Big/ \prod_{1\le u,v\le n}(x_u + y_v)} \\
&= (-1)^{j+i}\frac{\prod_{1\le u,v\le n}(x_u + y_v)}{\prod_{\substack{1\le u,v\le n\\ u\ne j\\ v\ne i}}(x_u + y_v)}\,\frac{\prod_{\substack{1\le u<v\le n\\ u\ne j\\ v\ne j}}(x_u - x_v)}{\prod_{1\le u<v\le n}(x_u - x_v)}\,\frac{\prod_{\substack{1\le u<v\le n\\ u\ne i\\ v\ne i}}(y_u - y_v)}{\prod_{1\le u<v\le n}(y_u - y_v)} \\
&= (-1)^{j+i}\prod_{\substack{1\le u,v\le n\\ u=j\vee v=i}}(x_u + y_v)\,\frac{1}{\prod_{\substack{1\le u<v\le n\\ u=j\vee v=j}}(x_u - x_v)}\,\frac{1}{\prod_{\substack{1\le u<v\le n\\ u=i\vee v=i}}(y_u - y_v)} \\
&= (-1)^{j+i}(x_j + y_i)\prod_{\substack{1\le u\le n\\ u\ne j}}(x_u + y_i)\prod_{\substack{1\le v\le n\\ v\ne i}}(x_j + y_v) \\
&\qquad \frac{1}{\left(\prod_{1\le u\le j-1}(x_u - x_j)\right)\left(\prod_{j+1\le v\le n}(x_j - x_v)\right)}\,\frac{1}{\left(\prod_{1\le u\le i-1}(y_u - y_i)\right)\left(\prod_{i+1\le v\le n}(y_i - y_v)\right)} \\
&= (-1)^{j+i}(x_j + y_i)\prod_{\substack{1\le u\le n\\ u\ne j}}(x_u + y_i)\prod_{\substack{1\le v\le n\\ v\ne i}}(x_j + y_v)\,\frac{(-1)^{j-1}}{\prod_{\substack{1\le u\le n\\ u\ne j}}(x_j - x_u)}\,\frac{(-1)^{i-1}}{\prod_{\substack{1\le v\le n\\ v\ne i}}(y_i - y_v)} \\
&= (-1)^{2(j+i-1)}(x_j + y_i)\frac{\prod_{\substack{1\le u\le n\\ u\ne j}}(x_u + y_i)}{\prod_{\substack{1\le u\le n\\ u\ne j}}(x_j - x_u)}\,\frac{\prod_{\substack{1\le v\le n\\ v\ne i}}(x_j + y_v)}{\prod_{\substack{1\le v\le n\\ v\ne i}}(y_i - y_v)} \\
&= \frac{(x_j + y_i)^2\left(\prod_{\substack{1\le u\le n\\ u\ne j}}(x_u + y_i)\right)\left(\prod_{\substack{1\le v\le n\\ v\ne i}}(x_j + y_v)\right)}{(x_j + y_i)\left(\prod_{\substack{1\le u\le n\\ u\ne j}}(x_j - x_u)\right)\left(\prod_{\substack{1\le v\le n\\ v\ne i}}(y_i - y_v)\right)} \\
&= \frac{\left(\prod_{1\le u\le n}(x_u + y_i)\right)\left(\prod_{1\le v\le n}(x_j + y_v)\right)}{(x_j + y_i)\left(\prod_{\substack{1\le u\le n\\ u\ne j}}(x_j - x_u)\right)\left(\prod_{\substack{1\le v\le n\\ v\ne i}}(y_i - y_v)\right)} \\
&= \frac{\prod_{1\le k\le n}(x_j + y_k)(x_k + y_i)}{(x_j + y_i)\left(\prod_{\substack{1\le k\le n\\ k\ne j}}(x_j - x_k)\right)\left(\prod_{\substack{1\le k\le n\\ k\ne i}}(y_i - y_k)\right)}
\end{aligned}
$$

as we needed to show.

42. [*M18*] What is the sum of all n^2 elements in the inverse of the combinatorial matrix?

For the inverse of the combinatorial matrix

$$\left[\frac{1}{y+\delta_{ij}x}\right]_n^{-1} = \left[\frac{-y+\delta_{ij}(x+ny)}{x(x+ny)}\right]_n$$

the sum of all n^2 elements is simply

$$\begin{aligned}\sum_{1\le i\le n}\sum_{1\le j\le n}\frac{-y+\delta_{ij}(x+ny)}{x(x+ny)} &= \sum_{1\le i\le n}\frac{\left(\sum_{\substack{1\le j\le n\\ j\ne i}} -y\right)-y+x+ny}{x(x+ny)}\\ &= \sum_{1\le i\le n}\frac{-y(n-1)-y+x+ny}{x(x+ny)}\\ &= \sum_{1\le i\le n}\frac{-ny+y-y+x+ny}{x(x+ny)}\\ &= \sum_{1\le i\le n}\frac{x}{x(x+ny)}\\ &= \sum_{1\le i\le n}\frac{1}{x+ny}\\ &= \frac{n}{x+ny}.\end{aligned}$$

43. [*M24*] What is the sum of all n^2 elements in the inverse of Vandermonde's matrix? [*Hint:* Use exercise 33.]

For the inverse of the Vandermonde matrix

$$\left[x_j^i\right]_n^{-1} = \left[\left(\sum_{\substack{1\le k_1<\ldots<k_{n-j}\le n\\ k_1,\ldots,k_{n-j}\ne i}} x_{k_1}\ldots x_{k_{n-j}}\right)\Bigg/ x_i\prod_{\substack{1\le k\le n\\ k\ne i}}(x_k-x_i)\right]_n = [b_{ij}]_n\,,$$

we want to find the sum of all n^2 elements $\sum_{1\le i,j\le n} b_{ij}$.

In the case that any $x_\kappa = 1$, $1\le\kappa\le n$, from exercise 40, we have that

$$[b_{ij}]_n[x_j^i]_n = \left[\sum_{1\le k\le n} b_{ik}x_j^k\right]_n = [\delta_{ij}]_n$$

or equivalently that

$$\begin{aligned}&\sum_{1\le j\le n} b_{ij} = \delta_{i\kappa}\\ &\implies \sum_{1\le i\le n}\sum_{1\le j\le n} b_{ij} = 1\\ &\implies \sum_{1\le i,j\le n} b_{ij} = 1-(1-1/x_\kappa)\\ &\implies \sum_{1\le i,j\le n} b_{ij} = 1-\prod_{1\le k\le n}(1-1/x_k).\end{aligned}$$

Otherwise, in the case that $x_\kappa \neq 1$, $1 \leq \kappa \leq n$, again from exercise 40, given $P_i(x) = \sum_{1\leq k\leq n} b_{ik}x^k$ for $x = 1$ we have that

$$\sum_{1\leq i,j\leq n} b_{ij} = \sum_{1\leq i\leq n} P_i(x) = \sum_{1\leq i\leq n} \frac{x}{x_i} \prod_{\substack{1\leq k\leq n\\ k\neq i}} \frac{x_k - x}{x_k - x_i}$$

or equivalently, for $x = 1$, that

$$\begin{aligned}
\sum_{1\leq i\leq n} P_i(1) &= \sum_{1\leq i\leq n} \left(\frac{1}{x_i} \prod_{\substack{1\leq k\leq n\\ k\neq i}} \frac{x_k - 1}{x_k - x_i} \right) \\
&= \sum_{1\leq i\leq n} \left(\frac{x_i - 1}{x_i - 1} \frac{1}{x_i} \prod_{\substack{1\leq k\leq n\\ k\neq i}} \frac{x_k - 1}{x_k - x_i} \right) \\
&= \sum_{1\leq i\leq n} \frac{(x_i - 1)\prod_{\substack{1\leq k\leq n\\ k\neq i}}(x_k - 1)}{x_i(x_i - 1)\prod_{\substack{1\leq k\leq n\\ k\neq i}}(x_k - x_i)} \\
&= \sum_{1\leq i\leq n} \frac{\prod_{1\leq k\leq n}(x_k - 1)}{x_i(x_i - 1)\prod_{\substack{1\leq k\leq n\\ k\neq i}}(x_k - x_i)} \\
&= \prod_{1\leq k\leq n} (x_k - 1) \sum_{1\leq i\leq n} \frac{1}{x_i(x_i - 1)\prod_{\substack{1\leq k\leq n\\ k\neq i}}(x_k - x_i)} \\
&= \prod_{1\leq k\leq n} (1 - x_k) \sum_{1\leq i\leq n} \frac{1}{x_i(x_i - 1)\prod_{\substack{1\leq k\leq n\\ k\neq i}}(x_i - x_k)} \\
&= \prod_{1\leq k\leq n} (x_k - 1) \left(\sum_{1\leq i\leq n} \frac{1}{x_i \prod_{\substack{1\leq k\leq n\\ k\neq i}}(x_i - x_k)} - \sum_{1\leq i\leq n} \frac{1}{(x_i - 1)\prod_{\substack{1\leq k\leq n\\ k\neq i}}(x_i - x_k)} \right).
\end{aligned}$$

Also, from exercise 33, we have for an extrapolated and distinct $x_0 = 1$ and for $r = 0$ that

$$\sum_{0\leq i\leq n} \left(\frac{1}{\prod_{\substack{0\leq j\leq n\\ j\neq i}}(x_i - x_j)} \right) = \frac{1}{\prod_{1\leq j\leq n}(1 - x_j)} + \sum_{1\leq i\leq n} \left(\frac{1}{(x_i - 1)\prod_{\substack{1\leq j\leq n\\ j\neq i}}(x_i - x_j)} \right) = 1$$

or equivalently that

$$\sum_{1\leq i\leq n} \left(\frac{1}{(x_i - 1)\prod_{\substack{1\leq j\leq n\\ j\neq i}}(x_i - x_j)} \right) = 1 - \frac{1}{\prod_{1\leq j\leq n}(1 - x_j)}.$$

Similarly, if $x_0 = 0$

$$\sum_{1\leq i\leq n} \left(\frac{1}{x_i \prod_{\substack{1\leq j\leq n\\ j\neq i}}(x_i - x_j)} \right) = 1 - \frac{1}{\prod_{1\leq j\leq n}(-x_j)}.$$

This yields

$$\begin{aligned}
\sum_{1\le i,j\le n} b_{ij} &= \sum_{1\le i\le n} P_i(1) \\
&= \prod_{1\le k\le n}(x_k-1)\left(\sum_{1\le i\le n}\frac{1}{x_i\prod_{\substack{1\le k\le n\\ k\ne i}}(x_i-x_k)} - \sum_{1\le i\le n}\frac{1}{(x_i-1)\prod_{\substack{1\le k\le n\\ k\ne i}}(x_i-x_k)}\right) \\
&= \prod_{1\le k\le n}(x_k-1)\left(\left(1-\frac{1}{\prod_{1\le j\le n}(-x_j)}\right)-\left(1-\frac{1}{\prod_{1\le j\le n}(1-x_j)}\right)\right) \\
&= \prod_{1\le k\le n}(x_k-1)\left(\frac{1}{\prod_{1\le j\le n}(-x_j)}-\frac{1}{\prod_{1\le j\le n}(1-x_j)}\right) \\
&= \frac{\prod_{1\le k\le n}(x_k-1)}{\prod_{1\le j\le n}(-x_j)}-\frac{\prod_{1\le k\le n}(x_k-1)}{\prod_{1\le j\le n}(1-x_j)} \\
&= \frac{\prod_{1\le k\le n}(1-x_k)}{\prod_{1\le j\le n}(1-x_j)}-\frac{\prod_{1\le k\le n}(x_k-1)}{\prod_{1\le j\le n}(x_j)} \\
&= 1-\frac{\prod_{1\le k\le n}(x_k-1)}{\prod_{1\le j\le n}(x_j)} \\
&= 1-\prod_{1\le k\le n}\frac{x_k-1}{x_k} \\
&= 1-\prod_{1\le k\le n}(1-1/x_k).
\end{aligned}$$

Therefore, in all cases,

$$\sum_{1\le i,j\le n} b_{ij} = 1-\prod_{1\le k\le n}(1-1/x_k).$$

► **44.** [*M26*] What is the sum of all n^2 elements in the inverse of Cauchy's matrix?

For the inverse of the Cauchy matrix

$$[a_{ij}]_n^{-1} = [b_{ij}]_n = \left[\frac{\prod_{1\le k\le n}(x_j+y_k)(x_k+y_i)}{(x_j+y_i)\left(\prod_{\substack{1\le k\le n\\ k\ne j}}(x_j-x_k)\right)\left(\prod_{\substack{1\le k\le n\\ k\ne i}}(y_i-y_k)\right)}\right]_n$$

we want to find the sum of all n^2 elements $\sum_{1\le i,j\le n} b_{ij}$.

But

$$\begin{aligned}
\sum_{1\le i\le n} b_{ij} &= \sum_{1\le i\le n} \frac{\left(\prod_{1\le k\le n}(x_j+y_k)(x_k+y_i)\right)}{(x_j+y_i)\left(\prod_{\substack{1\le k\le n\\ k\ne j}}(x_j-x_k)\right)\left(\prod_{\substack{1\le k\le n\\ k\ne i}}(y_i-y_k)\right)}\\
&= \sum_{1\le i\le n} \frac{\left(\prod_{1\le k\le n}(x_j+y_k)\right)\left(\prod_{1\le k\le n}(x_k+y_i)\right)}{(x_j+y_i)\left(\prod_{\substack{1\le k\le n\\ k\ne j}}(x_j-x_k)\right)\left(\prod_{\substack{1\le k\le n\\ k\ne i}}(y_i-y_k)\right)}\\
&= \frac{\prod_{1\le k\le n}(x_j+y_k)}{\prod_{\substack{1\le k\le n\\ k\ne j}}(x_j-x_k)} \sum_{1\le i\le n} \frac{\prod_{1\le k\le n}(x_k+y_i)}{(x_j+y_i)\prod_{\substack{1\le k\le n\\ k\ne i}}(y_i-y_k)}\\
&= \frac{\prod_{1\le k\le n}(x_j+y_k)}{\prod_{\substack{1\le k\le n\\ k\ne j}}(x_j-x_k)} \sum_{1\le i\le n} \frac{\prod_{\substack{1\le k\le n\\ k\ne i}}(x_k+y_i)}{\prod_{\substack{1\le k\le n\\ k\ne i}}(y_i-y_k)}\\
&= \frac{\prod_{1\le k\le n}(x_j+y_k)}{\prod_{\substack{1\le k\le n\\ k\ne j}}(x_j-x_k)}
\end{aligned}$$

since in general

$$\sum_{1\le i\le n} \frac{\prod_{1\le k\le n-1}(y_i-z_k)}{\prod_{\substack{1\le k\le n\\ k\ne i}}(y_i-y_k)} = 1$$

from exercise 34.

Then, for some polynomial $P(x)$ of order $n-2$ and by exercise 33

$$\begin{aligned}
\sum_{1\le i,j\le n} b_{ij} &= \sum_{1\le i\le n} \frac{\prod_{1\le k\le n}(x_j+y_k)}{\prod_{\substack{1\le k\le n\\ k\ne j}}(x_j-x_k)}\\
&= \sum_{1\le i\le n} \frac{x_j^n + \left(\sum_{1\le k\le n} y_k\right)x_j^{n-1} + P(x_j)}{\prod_{\substack{1\le k\le n\\ k\ne j}}(x_j-x_k)}\\
&= \sum_{1\le i\le n} \frac{x_j^n}{\prod_{\substack{1\le k\le n\\ k\ne j}}(x_j-x_k)} + \left(\sum_{1\le k\le n} y_k\right)\sum_{1\le i\le n} \frac{x_j^{n-1}}{\prod_{\substack{1\le k\le n\\ k\ne j}}(x_j-x_k)} + \sum_{1\le i\le n} \frac{P(x_j)}{\prod_{\substack{1\le k\le n\\ k\ne j}}(x_j-x_k)}\\
&= \sum_{1\le k\le n} x_k + \sum_{1\le k\le n} y_k + 0\\
&= \sum_{1\le k\le n} (x_k+y_k).
\end{aligned}$$

▶ **45.** [*M25*] A *Hilbert matrix*, sometimes called an $n\times n$ segment of *the* (infinite) Hilbert matrix, is a matrix for which $a_{ij} = 1/(i+j-1)$. Show that this is a special case of Cauchy's matrix, find its inverse, show that each element of the inverse is an integer, and show that the sum of all elements of the inverse is n^2. [*Note:* Hilbert matrices have often been used to test various matrix manipulation algorithms, because they are numerically unstable, and they have known inverses. However, it is a mistake to compare the *known* inverse, given in this exercise, to the *computed* inverse of a Hilbert matrix, since the matrix to be inverted must be expressed in rounded numbers beforehand; the inverse of an approximate Hilbert matrix will be somewhat different from the inverse of an exact one, due to the instability present. Since the elements of the inverse are integers, and since the inverse matrix is just as unstable as the original, the inverse can be specified exactly, and one should try to invert the inverse. The integers that appear in the inverse are, however, quite large.] The solution to this problem requires an elementary knowledge of factorials and binomial coefficients, which are discussed in Sections 1.2.5 and 1.2.6.

The Hilbert matrix, defined as

$$\mathbf{A}_n = [a_{ij}]_n = \left[\frac{1}{i+j-1}\right]_n = \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-1} \end{bmatrix}_n$$

is a special case of a Cauchy matrix $\mathbf{C}_n = [c_{ij}]_n = \left[\frac{1}{x_i+y_j}\right]_n$ with $x_i = i$ and $y_j = j-1$.

As such, its inverse $[b_{ij}]_n = \left[\frac{1}{i+j-1}\right]_n^{-1}$ has elements given by

$$\begin{aligned}
b_{ij} &= \frac{\prod_{1\le k\le n}(j+k-1)(k+i-1)}{(j+i-1)\left(\prod_{\substack{1\le k\le n\\ k\ne j}}(j-k)\right)\left(\prod_{\substack{1\le k\le n\\ k\ne i}}(i-k)\right)} \\
&= \frac{\prod_{1\le k\le n}(k+i-1)\prod_{1\le k\le n}(k+i-1)}{(j+i-1)\left(\prod_{\substack{1\le k\le j\\ k\ne j}}(j-k)\right)\left(\prod_{\substack{j\le k\le n\\ k\ne j}}(-1)(k-j)\right)\left(\prod_{\substack{1\le k\le i\\ k\ne i}}(i-k)\right)\left(\prod_{\substack{i\le k\le n\\ k\ne i}}(-1)(k-i)\right)} \\
&= \left(\frac{(j+n-1)!}{(j-1)!}\frac{(i+n-1)!}{(i-1)!}\right) \Big/ \left((j+i-1)\frac{(j-1)!}{1!}(-1)^{n-j+1}\frac{(n-j)!}{1!}\frac{(i-1)!}{1!}(-1)^{n-i+1}\frac{(n-i)!}{1!}\right) \\
&= \frac{(j+n-1)!(i+n-1)!}{(-1)^{2n-i-j+2}(j+i-1)(j-1)!(i-1)!(j-1)!(n-j)!(i-1)!(n-i)!} \\
&= \frac{(j+n-1)!(i+n-1)!}{(-1)^{-(i+j)}(j+i-1)((j-1)!(i-1)!)^2(n-j)!(n-i)!} \\
&= \frac{(-1)^{i+j}(i+n-1)!(j+n-1)!}{(i+j-1)(i-1)!^2(j-1)!^2(n-i)!(n-j)!}.
\end{aligned}$$

It is clear that b_{ij} is an integer, as it may be cast into binomial coefficients as

$$\begin{aligned}
b_{ij} &= \frac{(-1)^{i+j}(i+n-1)!(j+n-1)!}{(i+j-1)(i-1)!^2(j-1)!^2(n-i)!(n-j)!} \\
&= (-1)^{i+j}\frac{1}{(j-1)!(i-1)!}\frac{(i+n-1)!}{(i-1)!}\frac{(j+n-1)!}{(j+i-1)(n-i)!}\frac{1}{(n-j)!(j-1)!} \\
&= (-1)^{i+j}j\frac{(i+j-2)!}{(j-1)!(i-1)!}\frac{(i+n-1)!}{n!(i-1)!}\frac{(j+n-1)!}{(j+i-1)!(n-i)!}\frac{n!}{(n-j)!j!} \\
&= (-1)^{i+j}j\frac{(i+j-2)!}{((i+j-2)-(i-1))!(i-1)!}\frac{(i+n-1)!}{((i+n-1)-(i-1))!(i-1)!} \\
&\qquad \frac{(j+n-1)!}{((j+n-1)-(n-i))!(n-i)!}\frac{n!}{(n-j)!j!} \\
&= (-1)^{i+j}j\binom{i+j-2}{i-1}\binom{i+n-1}{i-1}\binom{j+n-1}{n-i}\binom{n}{j}.
\end{aligned}$$

From exercise 44, the sum of the elements of the inverse is simply

$$\begin{aligned}\sum_{1\le i,j\le n} b_{ij} &= \sum_{1\le k\le n} (k+k-1)\\ &= 2\left(\sum_{1\le k\le n} k\right) - \sum_{1\le k\le n} 1\\ &= 2\frac{n(n+1)}{2} - n\\ &= n^2+n-n\\ &= n^2.\end{aligned}$$

[For further information, see J. Todd, *J. Research Nat. Bur. Stand.* **65** (1961), 19–22; A. Cauchy, *Exercices d'analyse et de physique mathématique* **2** (1841), 151–159.]

► **46.** [*M30*] Let A be an $m \times n$ matrix, and let B be an $n \times m$ matrix. Given that $1 \le j_1, j_2, \ldots, j_m \le n$, let $A_{j_1 j_2 \ldots j_m}$ denote the $m \times m$ matrix consisting of columns $j_1, \ldots, j_m$ of A, and let $B_{j_1 j_2 \ldots j_m}$ denote the $m \times m$ matrix consisting of rows $j_1, \ldots, j_m$ of B. Prove the *Binet-Cauchy identity*

$$\det(AB) = \sum_{1\le j_1<j_2<\cdots<j_m\le n} \det(A_{j_1 j_2 \ldots j_m}) \det(B_{j_1 j_2 \ldots j_m}).$$

(Note the special cases: (i) $m = n$, (ii) $m = 1$, (iii) $B = A^T$, (iv) $m > n$, (v) $m = 2$.)

Proposition. $\det(AB) = \sum_{1\le j_1<j_2<\cdots<j_m\le n} \det(A_{j_1 j_2 \ldots j_m}) \det(B_{j_1 j_2 \ldots j_m})$.

Proof. Let A be an $m \times n$ matrix, and let B be an $n \times m$ matrix. Given that $1 \le j_1, j_2, \ldots, j_m \le n$, let $A_{j_1 j_2 \ldots j_m}$ denote the $m\times m$ matrix consisting of columns $j_1, \ldots, j_m$ of A, and let $B_{j_1 j_2 \ldots j_m}$ denote the $m \times m$ matrix consisting of rows $j_1, \ldots, j_m$ of B. We must show that

$$\det(AB) = \sum_{1\le j_1<j_2<\cdots<j_m\le n} \det(A_{j_1 j_2 \ldots j_m}) \det(B_{j_1 j_2 \ldots j_m}).$$

Let

$$\epsilon(k_1, \ldots, k_m) = \operatorname{sign}(\prod_{1\le i<j\le m} (k_j - k_i))$$

for

$$\operatorname{sign}(x) = [x > 0] - [x < 0]$$

be the *Levi-Civita* function so that in general

$$\det([c_i j]_n) = \sum_{1\le i_1,\ldots,i_n\le n} \epsilon(i_1, \ldots, i_n) \prod_{1\le i\le n} a_{i,i_i};$$

and so that if $(k_1, \ldots, k_m)$ and $(l_1, \ldots, l_m)$ are identical except that $k_i = l_j$ and $k_j = l_i$, so that $\epsilon(k_1, \ldots, k_m)$ and $-\epsilon(l_1, \ldots, l_m)$, then in general

$$\det(B_{k_1\ldots k_m}) = \epsilon(k_1, \ldots, k_m) \det(B_{j_1\ldots j_m})$$

if $j_1 \le \cdots \le j_m$ are the numbers $k_1, \ldots, k_m$ rearranged into nondecreasing order.

Then

$$\begin{aligned}\det(AB) &= \sum_{1\le l_1,\ldots,l_m\le m} \epsilon(l_1,\ldots,l_m)\left(\sum_{1\le k\le n} a_{1k}b_{kl_1}\right)\cdots\left(\sum_{1\le k\le n} a_{mk}b_{kl_m}\right)\\ &= \sum_{1\le k_1,\ldots,k_m\le n} a_{1k_1}\ldots a_{mk_m} \sum_{1\le l_1,\ldots,l_m\le m} \epsilon(l_1,\ldots,l_m) b_{k_1l_1}\ldots b_{k_ml_m}\\ &= \sum_{1\le k_1,\ldots,k_m\le n} a_{1k_1}\ldots a_{mk_m}\det(B_{k_1\ldots k_m})\\ &= \sum_{1\le k_1,\ldots,k_m\le n} \epsilon(k_1,\ldots,k_m) a_{1k_1}\ldots a_{mk_m}\det(B_{j_1\ldots j_m})\\ &= \sum_{1\le j_1\le\cdots\le j_m\le n} \det(A_{j_1\ldots j_m})\det(B_{j_1\ldots j_m})\end{aligned}$$

as we needed to show.

Note that if $m = n$, we have the usual identity for square matrices

$$\det(AB) = \det(A)\det(B);$$

if $m = 1$, we have the dot product

$$\det(AB) = \sum_{1\le k\le n} a_{1k}b_{k1} = A\cdot B;$$

if $B = A^T$, we have the square

$$\det(AB) = \det(AA^T) = \sum_{1\le j_1\le\cdots\le j_m\le n} \det(A_{j_1\ldots j_m})^2;$$

and if $m = 2$, we have the first nontrivial case of the identity

$$\det(AB) = \sum_{1\le j_1\le j_2\le n} \det(A_{j_1j_2})\det(B_{j_1j_2}).$$

□

[*J. de l'École Polytechnique* **9** (1813), 280–354; **10** (1815), 29–112. Binet and Cauchy presented their papers on the same day in 1812.]

47. [*M27*] (C. Krattenthaler.) Prove that

$$\det\begin{pmatrix}(x+q_2)(x+q_3) & (x+p_1)(x+q_3) & (x+p_1)(x+p_2)\\ (y+q_2)(y+q_3) & (y+p_1)(y+q_3) & (y+p_1)(y+p_2)\\ (z+q_2)(z+q_3) & (z+p_1)(z+q_3) & (z+p_1)(z+p_2)\end{pmatrix}$$
$$= (x-y)(x-z)(y-z)(p_1-q_2)(p_1-q_3)(p_2-q_3).$$

and generalize this equation to an identity for an $n\times n$ determinant in $3n-2$ variables $x_1,\ldots,x_n,p_1,\ldots,p_{n-1},q_2,\ldots,q_n$. Compare your formula to the result of exercise 38.

We may prove the more general equation.

Proposition. $\det\left[\left(\prod_{1\le k\le j-1}(x_i+p_k)\right)\left(\prod_{j+1\le k\le n}(x_i+q_k)\right)\right]_n = \prod_{1\le i<j\le n}(x_i - x_j)(p_i - q_j)$.

Proof. Let

$$
\begin{aligned}
\mathbf{A}_n &= [a_{ij}]_n \\
&= \left[\left(\prod_{1\le k\le j-1}(x_i+p_k)\right)\left(\prod_{j+1\le k\le n}(x_i+q_k)\right)\right]_n \\
&= \begin{bmatrix}
\prod_{2\le k\le n}(x_1+q_k) & (x_1+p_1)\prod_{3\le k\le n}(x_1+q_k) & \dots & \prod_{1\le k\le n-1}(x_1+p_k) \\
\prod_{2\le k\le n}(x_2+q_k) & (x_2+p_1)\prod_{3\le k\le n}(x_2+q_k) & \dots & \prod_{1\le k\le n-1}(x_2+p_k) \\
\vdots & \vdots & \ddots & \vdots \\
\prod_{2\le k\le n}(x_n+q_k) & (x_n+p_1)\prod_{3\le k\le n}(x_n+q_k) & \dots & \prod_{1\le k\le n-1}(x_n+p_k)
\end{bmatrix}_n .
\end{aligned}
$$

We must show that

$$\det[a_{ij}]_n = \prod_{1\le i<j\le n}(x_i-x_j)(p_i-q_j).$$

Let

$$a'_{ij} = \begin{cases} a_{i1} & \text{if } j=1 \\ a_{ij}-a_{i,j-1} & \text{if } 2\le j\le n \end{cases}$$

so that $\det[a_{ij}]_n = \det[a'_{ij}]_n$ where

$$a'_{ij} = \begin{cases} \left(\prod_{1\le k\le j-1}(x_i+p_k)\right)\left(\prod_{j+1\le k\le n}(x_i+q_k)\right) & \text{if } j=1 \\ (p_{j-1}-q_j)\left(\prod_{1\le k\le j-2}(x_i+p_k)\right)\left(\prod_{j+1\le k\le n}(x_i+q_k)\right) & \text{if } 2\le j\le n \end{cases}$$

since for $2\le j\le n$

$$
\begin{aligned}
a'_{ij} &= \left(\prod_{1\le k\le j-1}(x_i+p_k)\right)\left(\prod_{j+1\le k\le n}(x_i+q_k)\right) - \left(\prod_{1\le k\le j-2}(x_i+p_k)\right)\left(\prod_{j\le k\le n}(x_i+q_k)\right) \\
&= \left((x_i+p_{j-1})\prod_{1\le k\le j-2}(x_i+p_k)\right)\left(\prod_{j+1\le k\le n}(x_i+q_k)\right) \\
&\quad - \left(\prod_{1\le k\le j-2}(x_i+p_k)\right)\left((x_i+q_j)\prod_{j+1\le k\le n}(x_i+q_k)\right) \\
&= ((x_i+p_{j-1})-(x_i+q_j))\left(\prod_{1\le k\le j-2}(x_i+p_k)\right)\left(\prod_{j+1\le k\le n}(x_i+q_k)\right) \\
&= (p_{j-1}-q_j)\left(\prod_{1\le k\le j-2}(x_i+p_k)\right)\left(\prod_{j+1\le k\le n}(x_i+q_k)\right).
\end{aligned}
$$

Let

$$q_{ij} = \begin{cases} 1 & \text{if } j=1 \\ p_{j-1}-q_j & \text{if } 2\le j\le n \end{cases}$$

so that

$$[a'_{ij}]_n = ([q_{ij}]_n[b_{ij}]_n^T)^T$$

and that

$$\begin{aligned}\det[a'_{ij}]_n &= \det(([q_{ij}]_n[b_{ij}]_n^T)^T)\\ &= \det([q_{ij}]_n[b_{ij}]_n^T)\\ &= \det[q_{ij}]_n \det([b_{ij}]_n^T)\\ &= \prod_{2\le k\le n}(p_{k-1}-q_k)\det([b_{ij}]_n^T)\\ &= \prod_{2\le k\le n}(p_{k-1}-q_k)\det[b_{ij}]_n\end{aligned}$$

where

$$[b_{ij}] = \left[\left(\prod_{1\le k\le j-2}(x_i+p_k)\right)\left(\prod_{j+1\le k\le n}(x_i+q_k)\right)\right]_n.$$

Repeating this transformation, each time over fewer columns to factor out $(p_{j-2}-q_{j-1})$, $(p_{j-3}-q_{j-2})$, etc., we eventually have

$$\det[a_{ij}]_n = \prod_{1\le i<j\le n}(p_i-q_j)\det[b_{ij}]_n$$

for

$$[b_{ij}]_n = \left[\prod_{j+1\le k\le n}(x_i+q_k)\right]_n.$$

Then let

$$b'_{ij} = \begin{cases} b_{ij}-(q_{j+1})b_{i,j+1} & \text{if } 1\le j\le n-1\\ b_{in} & \text{if } j=n\end{cases}$$

so that $\det[b_{ij}]_n = \det[b'_{ij}]_n$ where

$$b'_{ij} = \begin{cases} x_i\prod_{j+2\le k\le n}(x_i+q_k) & \text{if } 1\le j\le n-1\\ x_i & \text{if } j=n\end{cases}$$

since for $1\le j\le n-1$

$$\begin{aligned}b'_{ij} &= \prod_{j+1\le k\le n}(x_i+q_k)-(q_{j+1})\prod_{j+2\le k\le n}(x_i+q_k)\\ &= (x_i+q_{j+1})\prod_{j+2\le k\le n}(x_i+q_k)-(q_{j+1})\prod_{j+2\le k\le n}(x_i+q_k)\\ &= ((x_i+q_{j+1})-(q_{j+1}))\prod_{j+2\le k\le n}(x_i+q_k)\\ &= x_i\prod_{j+2\le k\le n}(x_i+q_k).\end{aligned}$$

Repeating this transformation also, each time over fewer columns to factor out q_{j+2}, q_{j+3}, etc., we eventually have

$$\det[a_{ij}]_n = \prod_{1\le i<j\le n}(p_i-q_j)\det[b_{ij}]_n = \prod_{1\le i<j\le n}(p_i-q_j)\det[b'_{ij}]_n$$

for

$$[b'_{ij}]_n = \left[x_i^{n-j}\right]_n.$$

Let

$$c'_{ij} = \begin{cases} b'_{1j} & \text{if } i = 1 \\ b'_{ij} - b'_{1j} & \text{if } 2 \le i \le n \end{cases}$$

and

$$c''_{ij} = \begin{cases} c'_{1j} - x_1 c'_{1(j+1)} & \text{if } i = 1, 1 \le j \le n-1 \\ c'_{1n} & \text{if } i = 1, j = n \\ c'_{ij} - x_1 c'_{i(j+1)} & \text{if } 2 \le i \le n, 1 \le j \le n-1 \\ c'_{in} & \text{if } 2 \le i \le n, j = n \le n, \end{cases}$$

so that $\det[b'_{ij}] = \det[c''_{ij}]$ where

$$c''_{ij} = \begin{cases} 0 & \text{if } i = 1, 1 \le j \le n-1 \\ 1 & \text{if } i = 1, j = n \\ x_i^{n-j-1}(x_i - x_1) & \text{if } 2 \le i \le n, 1 \le j \le n-1 \\ 0 & \text{if } 2 \le i \le n, j = n \le n, \end{cases}$$

since:

$$\begin{array}{ll}
c'_{1j} = b'_{1j} = x_1^{n-j} & i = 1, 1 \le j \le n-1 \\
c'_{1n} = b'_{1n} = x_1^0 = 1 & i = 1, j = n \\
c'_{ij} = b'_{ij} - b'_{1j} = x_i^{n-j} - x_1^{n-j} & 2 \le i \le n, 1 \le j \le n-1 \\
c'_{in} = b'_{ij} - b'_{1j} = x_i^0 - x_1^0 = 0 & 2 \le i \le n, j = n \le n \\
c''_{1j} = x_1^{n-j} - x_1 x_1^{n-j-1} = 0 & i = 1, 1 \le j \le n-1 \\
c''_{1n} = x_1^0 = 1 & i = 1, j = n \\
c''_{ij} = x_i^{n-j} - x_1^{n-j} - x_1(x_i^{n-j-1} - x_1^{n-j-1}) = x_i^{n-j-1}(x_i - x_1) & 2 \le i \le n, 1 \le j \le n-1 \\
c''_{in} = x_i^0 - x_1^0 = 0 & 2 \le i \le n, j = n \le n
\end{array}$$

Let

$$r_{ij} = \begin{cases} x_{i+1} - x_1 & \text{if } i = j, 1 \le i, j \le n-1 \\ 0 & \text{otherwise} \end{cases}$$

so that $\operatorname{minor}([b']_n, 1, n) = [r_{ij}]_{n-1} \operatorname{minor}([c'']_n, 1, n)$ and $\det[r_{ij}]_{n-1} = \prod_{1 \le k \le n-1}(x_{k+1} - x_1)$. Then

$$\begin{aligned}
\det[c''_{ij}]_n &= \sum_{1 \le i \le n} c''_{in} \operatorname{cofactor}(c''_{in}) \\
&= c''_{1n} \operatorname{cofactor}(c''_{1n}) + \sum_{2 \le i \le n} c''_{in} \operatorname{cofactor}(c''_{in}) \\
&= (1)(-1)^{1+n} \det \operatorname{minor}([c'']_n, 1, n) + 0 \\
&= (-1)^{n+1} \det \operatorname{minor}([c'']_n, 1, n) \\
&= (-1)^{n+1} \det[r_{ij}]_{n-1} \det(\operatorname{minor}([b']_n, 1, n)) \\
&= (-1)^{n+1} \det[r_{ij}]_{n-1} \det \operatorname{minor}([b']_n, 1, n) \\
&= (-1)^{n+1} \left(\prod_{1 \le k \le n-1} (x_{k+1} - x_1) \right) \det \operatorname{minor}([b']_n, 1, n) \\
&= \left(\prod_{1 \le k \le n-1} (x_{k+1} - x_1) \right) \det \operatorname{minor}([b']_n, 1, n)
\end{aligned}$$

since a product of $n+1$ signs will cancel with $n-1$ (i.e., $(-1)^{n+1}(-1)^{n-1} = (-1)^{2n} = 1$). This recursive identity allows us to prove by mathematical induction on n that

$$\det[b'_{ij}]_k = \prod_{1\le i<j\le k} (x_i - x_j).$$

Therefore

$$\begin{aligned}
\det \mathbf{A}_n &= \det[a_{ij}]_n \\
&= \det\left[\left(\prod_{1\le k\le j-1}(x_i + p_k)\right)\left(\prod_{j+1\le k\le n}(x_i + q_k)\right)\right]_n \\
&= \prod_{1\le i<j\le n}(p_i - q_j)\det[b_{ij}]_n \\
&= \prod_{1\le i<j\le n}(p_i - q_j)\det[b'_{ij}]_n \\
&= \prod_{1\le i<j\le n}(p_i - q_j)\left[x_i^{n-j}\right]_n \\
&= \prod_{1\le i<j\le n}(p_i - q_j)\prod_{1\le i<j\le k}(x_i - x_j) \\
&= \prod_{1\le i<j\le n}(x_i - x_j)(p_i - q_j)
\end{aligned}$$

as we needed to show. □

[*Manuscripta Math.* **69** (1990), 177–178.]