Exercises from Section 1.2.5

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1. [*00*] How many ways are there to shuffle a 52-card deck?

As we have 52 choices for the first card, 51 for the second, and so on, we simply have 52! ways to shuffle a 52-card deck. 52! is the 68 decimal digit number

$$80658175170943878571660636856403766975289505440883277824000000000000.$$

2. [*10*] In the notation of Eq. (2), show that $p_{n(n-1)} = p_{nn}$, and explain why this happens.

In the notation of Eq. (2), since

$$p_{nk} = \prod_{n-k+1 \le j \le n} j$$

we have that

$$p_{nn} = \prod_{1 \le j \le n} j = \prod_{2 \le j \le n} j = \prod_{n-(n-1)+1 \le j \le n} j = p_{n(n-1)}.$$

That is, after choosing the $(n-1)$th element, we have no choice left for the last element.

3. [*10*] What permutations of $\{1, 2, 3, 4, 5\}$ would be constructed from the permutation 3 1 2 4 using Methods 1 and 2, respectively?

We can construct permutations of the set $\{1, 2, 3, 4, 5\}$ from the permutation 3 1 2 4 using either method.

In Method 1, we insert 5 in all possible positions to obtain

5 3 1 2 4, 3 5 1 2 4, 3 1 5 2 4, 3 1 2 5 4, and 3 1 2 4 5.

In Method 2, we start with an intermediary set of permutations

3 1 2 4 $\frac{1}{2}$, 3 1 2 4 $\frac{3}{2}$, 3 1 2 4 $\frac{5}{2}$, 3 1 2 4 $\frac{7}{2}$, and 3 1 2 4 $\frac{9}{2}$,

which are finally renamed as

4 2 3 5 1, 4 1 3 5 2, 4 1 2 5 3, 3 1 2 5 4, and 3 1 2 4 5.

► **4.** [*13*] Given the fact that $\log_{10} 1000! = 2567.60464\ldots$, determine exactly how many decimal digits are present in the number 1000!. What is the *most significant* digit? What is the *least significant* digit?

Since the number of decimal digits in a number n is given by $\lfloor \log_{10} n \rfloor + 1$, $\log_{10} 1000! = 2567.60464\ldots$ implies that 1000! has 2568 decimal digits.

The *most significant* digit is given by $\left\lfloor \frac{1000!}{10^{2568-1}} \right\rfloor$, but since $\log_{10} 4 = 0.60206\ldots$ and $\log_{10} 5 =$

$0.69897\ldots$,

$$\begin{aligned}
\log_{10} 1000! = 2567.60464\ldots \quad &\Longrightarrow \quad 2567 + \log_{10} 4 \le \log_{10} 1000! < 2567 + \log_{10} 5 \\
&\Longrightarrow \quad \log_{10} 10^{2567} + \log_{10} 4 \le \log_{10} 1000! < \log_{10} 10^{2567} + \log_{10} 5 \\
&\Longrightarrow \quad \log_{10}\left(4 \cdot 10^{2567}\right) \le \log_{10} 1000! < \log_{10}\left(5 \cdot 10^{2567}\right) \\
&\Longrightarrow \quad \log_{10} 4 \le \log_{10} \frac{1000!}{10^{2567}} < \log_{10} 5 \\
&\Longrightarrow \quad 4 \le \frac{1000!}{10^{2568-1}} < 5,
\end{aligned}$$

and so, the most significant digit is $\left\lfloor \frac{1000!}{10^{2568-1}} \right\rfloor = 4$.

The *least significant* digit is intuitively 0, since the factorial is some number multiplied by a factor of 1000. We can determine precisely how many zeros using Eq. (8), since

$$\begin{aligned}
\mu_2 &= \sum_{k>0} \left\lfloor \frac{1000}{2^k} \right\rfloor \\
&= \sum_{1 \le k \le 9} \left\lfloor \frac{1000}{2^k} \right\rfloor \\
&= 500 + 250 + 125 + 62 + 31 + 15 + 7 + 3 + 1 \\
&= 994
\end{aligned}$$

and

$$\begin{aligned}
\mu_5 &= \sum_{k>0} \left\lfloor \frac{1000}{5^k} \right\rfloor \\
&= \sum_{1 \le k \le 4} \left\lfloor \frac{1000}{5^k} \right\rfloor \\
&= 200 + 40 + 8 + 1 \\
&= 249
\end{aligned}$$

giving us that for some arbitrary integer z not divisible by 10, $1000! = 2^{994} \cdot 5^{249} z = 2^{745} z \cdot 10^{249}$; that is, $1000!$ is a number that ends with 249 zeros.

[*Scripta Mathematica* **21** (1955), 266–267]

5. [*15*] Estimate 8! using the more exact version of Stirling's approximation:

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n}\right).$$

We may estimate 8! as

$$\begin{aligned}
8! &\approx \sqrt{2\pi 8} \left(\frac{8}{e}\right)^8 \left(1 + \frac{1}{12(8)}\right) \\
&= 4\sqrt{\pi} \frac{16777216}{e^8} \frac{97}{96} \\
&= \frac{203423744\sqrt{\pi}}{3e^8} \\
&\approx 40318.
\end{aligned}$$

► **6.** [*17*] Using Eq. (8), write 20! as a product of prime factors.

Since the primes that divide 20 are 2, 3, 5, 7, 11, 13, 17, and 19, we may determine the multiplicity of the prime factors of 20! as

$$\begin{aligned}\mu_2 &= \sum_{k>0} \left\lfloor \frac{20}{2^k} \right\rfloor \\ &= \sum_{1\le k\le 4} \left\lfloor \frac{20}{2^k} \right\rfloor \\ &= 10+5+2+1 \\ &= 18,\end{aligned}$$

$$\begin{aligned}\mu_3 &= \sum_{k>0} \left\lfloor \frac{20}{2^k} \right\rfloor \\ &= \sum_{1\le k\le 2} \left\lfloor \frac{20}{2^k} \right\rfloor \\ &= 6+2 \\ &= 8,\end{aligned}$$

$$\begin{aligned}\mu_5 &= \sum_{k>0} \left\lfloor \frac{20}{5^k} \right\rfloor \\ &= \sum_{1\le k\le 1} \left\lfloor \frac{20}{5^k} \right\rfloor \\ &= 4,\end{aligned}$$

$$\begin{aligned}\mu_7 &= \sum_{k>0} \left\lfloor \frac{20}{7^k} \right\rfloor \\ &= \sum_{1\le k\le 1} \left\lfloor \frac{20}{7^k} \right\rfloor \\ &= 2,\end{aligned}$$

$$\begin{aligned}\mu_{11} &= \sum_{k>0} \left\lfloor \frac{20}{11^k} \right\rfloor \\ &= \sum_{1\le k\le 1} \left\lfloor \frac{20}{11^k} \right\rfloor \\ &= 1,\end{aligned}$$

$$\mu_{13} = \sum_{k>0} \left\lfloor \frac{20}{13^k} \right\rfloor = \sum_{1 \le k \le 1} \left\lfloor \frac{20}{13^k} \right\rfloor = 1,$$

$$\mu_{17} = \sum_{k>0} \left\lfloor \frac{20}{17^k} \right\rfloor = \sum_{1 \le k \le 1} \left\lfloor \frac{20}{17^k} \right\rfloor = 1,$$

and

$$\mu_{19} = \sum_{k>0} \left\lfloor \frac{20}{19^k} \right\rfloor = \sum_{1 \le k \le 1} \left\lfloor \frac{20}{19^k} \right\rfloor = 1,$$

so that $20! = 2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$.

7. [*M10*] Show that the "generalized terminal" function in Eq. (10) satisfies the identity $x? = x + (x-1)?$ for all real numbers x.

Proposition. $x? = x + (x-1)?$ *for all real numbers* x.

Proof. Let x be an arbitrary real number. We must show that

$$x? = x + (x-1)?.$$

But by Eq. (10)

$$\begin{aligned} x? &= \frac{1}{2}x(x+1) \\ &= x - x + \frac{1}{2}x(x+1) \\ &= x + \frac{-2x}{2} + \frac{1}{2}x(x+1) \\ &= x + \frac{1}{2}(x^2 + x - 2x) \\ &= x + \frac{1}{2}(x^2 - x) \\ &= x + \frac{1}{2}(x-1)x \\ &= x + \frac{1}{2}(x-1)((x-1)+1) \\ &= x + (x-1)? \end{aligned}$$

as we needed to show. □

8. [*HM15*] Show that the limit in Eq. (13) does equal $n!$ when n is a nonnegative integer.

When n is a nonnegative integer, we have

$$\begin{aligned}\lim_{m\to\infty}\frac{m^n m!}{\prod_{1\le k\le m}(n+k)} &= \lim_{m\to\infty}\frac{m^n m!}{(n+m)!/n!}\\ &= \lim_{m\to\infty}\frac{n!m^n m!}{(m+n)!}\\ &= \lim_{m\to\infty}\frac{n!m^n}{(m+n)!/m!}\\ &= \lim_{m\to\infty}\frac{n!m^n}{\prod_{1\le k\le n}(m+k)}\\ &= \lim_{m\to\infty} n!\prod_{1\le k\le n}\frac{m}{m+k}\\ &= n!\lim_{m\to\infty}\prod_{1\le k\le n}\frac{m}{m+k}\\ &= n!.\end{aligned}$$

9. [*M10*] Determine the values of $\Gamma(\frac{1}{2})$ and $\Gamma(-\frac{1}{2})$, given that $\left(\frac{1}{2}\right)! = \sqrt{\pi}/2$.

Given that $\left(\frac{1}{2}\right)! = \sqrt{\pi}/2$, we have that

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \left(\frac{1}{2}\right)!/\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{\pi}}{2}\frac{2}{1}\\ &= \sqrt{\pi}\end{aligned}$$

and that

$$\begin{aligned}\Gamma\left(\frac{-1}{2}\right) &= \frac{2}{-1}\Gamma\left(\frac{1}{2}\right)\\ &= -2\sqrt{\pi}\end{aligned}$$

since $\Gamma(n+1) = n\Gamma(n)$.

▶ **10.** [*HM20*] Does the identity $\Gamma(x+1) = x\Gamma(x)$ hold for all real numbers x? (See exercise 7.)

The identity $\Gamma(x+1) = x\Gamma(x)$ holds for all real numbers x, except when x is zero or a negative integer, since

$$\begin{aligned}\Gamma(x+1) &= \lim_{m\to\infty}\frac{m^x m!}{\prod_{1\le k\le m}(x+k)}\\ &= \lim_{m\to\infty}\frac{mx(x+m)}{x(x+m)}\frac{m^{x-1}m!}{\prod_{2\le k\le m+1}((x-1)+k)}\\ &= \lim_{m\to\infty}\frac{mx}{x+m}\frac{m^{x-1}m!}{\prod_{1\le k\le m}((x-1)+k)}\\ &= x\lim_{m\to\infty}\frac{m}{m+x}\frac{m^{x-1}m!}{\prod_{1\le k\le m}((x-1)+k)}\\ &= x\lim_{m\to\infty}\frac{m^{x-1}m!}{\prod_{1\le k\le m}((x-1)+k)}\\ &= x\Gamma(x).\end{aligned}$$

11. [*M15*] Let the representation of n in the binary system be $n = 2^{e_1} + 2^{e_2} + \cdots + 2^{e_r}$, where $e_1 > e_2 > \cdots > e_r \geq 0$. Show that $n!$ is divisible by 2^{n-r} but not by 2^{n-r+1}.

Given $n = \sum_{1 \leq j \leq r} 2^{e_j}$, we may find the exact multiplicity of the prime factor 2 from Eq. (8) as:

$$\begin{aligned}
\mu &= \sum_{i>0} \left\lfloor \frac{n}{2^i} \right\rfloor \\
&= \sum_{1 \leq i \leq r} \left\lfloor \frac{\sum_{1 \leq j \leq r} 2^{e_j}}{2^{e_i}} \right\rfloor \\
&= \sum_{1 \leq i \leq r} \left\lfloor \sum_{0 \leq j \leq r} \frac{2^{e_j}}{2^{e_i}} \right\rfloor \\
&= \sum_{1 \leq j \leq r} \sum_{1 \leq i \leq j} \frac{2^{e_j}}{2^{e_i}} \\
&= \sum_{1 \leq j \leq r} 2^{e_j} \sum_{1 \leq i \leq j} \frac{1}{2^{e_i}} \\
&= \sum_{1 \leq j \leq r} 2^{e_j} \frac{2^{-e_j}(2^{e_j} - 1)}{2 - 1} \\
&= \sum_{1 \leq j \leq r} \frac{2^{e_j}}{2^{e_j}} (2^{e_j} - 1) \\
&= \sum_{1 \leq j \leq r} (2^{e_j} - 1) \\
&= \sum_{1 \leq j \leq r} 2^{e_j} - \sum_{0 \leq j \leq r} 1 \\
&= n - \sum_{1 \leq j \leq r} 1 \\
&= n - r.
\end{aligned}$$

That is, $n!$ is divisible by 2^{n-1}, but not by 2^{n-r+1}.

► **12.** [*M22*] (A. Legendre, 1808.) Generalizing the result of the previous exercise, let p be a prime number, and let the representation of n in the p-ary number system be $n = a_k p^k + a_{k-1} p^{k-1} + \cdots + a_1 p + a_0$. Express the number μ of Eq. (8) in a simple formula involving n, p, and a's.

Given $n = \sum_{0 \le j \le k} a_j p^j$, we may express the number μ from Eq. (8) as:

$$\begin{aligned}
\mu &= \sum_{i>0} \left\lfloor \frac{n}{p^i} \right\rfloor \\
&= \sum_{1 \le i \le k} \left\lfloor \frac{\sum_{0 \le j \le k} a_j p^j}{p^i} \right\rfloor \\
&= \sum_{1 \le i \le k} \left\lfloor \sum_{0 \le j \le k} \frac{a_j p^j}{p^i} \right\rfloor \\
&= \sum_{0 \le j \le k} \sum_{1 \le i \le j} \frac{a_j p^j}{p^i} \\
&= \sum_{0 \le j \le k} a_j p^j \sum_{1 \le i \le j} \frac{1}{p^i} \\
&= \sum_{0 \le j \le k} a_j p^j \frac{p^{-j}(p^j - 1)}{p - 1} \\
&= \sum_{0 \le j \le k} a_j \frac{p^j}{p^j} \frac{p^j - 1}{p - 1} \\
&= \sum_{0 \le j \le k} a_j \frac{p^j - 1}{p - 1} \\
&= \frac{\sum_{0 \le j \le k} a_j (p^j - 1)}{p - 1} \\
&= \frac{\sum_{0 \le j \le k} a_j p^j - \sum_{0 \le j \le k} a_j}{p - 1} \\
&= \frac{n - \sum_{0 \le j \le k} a_j}{p - 1}.
\end{aligned}$$

13. [*M23*] (*Wilson's theorem*, actually due to Leibniz, 1682.) If p is prime, then $(p-1)! \bmod p = p - 1$. Prove this, by pairing off numbers among $\{1, 2, \ldots, p-1\}$ whose product modulo p is 1.

Proposition. *If p is prime,* $(p-1)! \bmod p = p - 1$.

Proof. Let p be an arbitrary prime. We must show that

$$(p-1)! \bmod p = p - 1.$$

By exercise 1.2.4-19, for each integer k, $1 < k < p - 1$, there is another integer k' such that

$$kk' \equiv 1 \pmod{p}$$

allowing us to 'pair off numbers' as

$$(p-2)!/1! \equiv 1 \pmod{p}.$$

Also, clearly,

$$1 \equiv 1 \pmod{p}$$

and

$$(p-1) \equiv p - 1 \pmod{p}.$$

And so,

$$(p-1)! \equiv p - 1 \pmod{p}$$

or equivalently

$$(p-1)! \bmod p = p-1$$

as we needed to show. □

► **14.** [*M28*] (L. Stickelberger, 1890.) In the notation of exercise 12, we can determine $n! \bmod p$ in terms of the p-ary representation, for *any* positive integer n, thus generalizing Wilson's theorem. In fact, prove that $n!/p^\mu \equiv (-1)^\mu a_0! a_1! \ldots a_k!$ (modulo p).

Proposition. $n!/p^\mu \equiv (-1)^\mu \prod_{0\le i\le k} a_i! \pmod p$.

Proof. Let n and p be arbitrary positive integers such that p is prime, $n = \sum_{0\le i\le k} a_i p^i$, and $\mu = \sum_{1\le j\le k} \left\lfloor \frac{n}{p^j} \right\rfloor$. We must show that

$$n!/p^\mu \equiv (-1)^\mu \prod_{0\le i\le k} a_i! \pmod p.$$

First consider the trivial case $k=0$, so that $n = a_0$ and $\mu = 0$. Then clearly

$$\begin{aligned} a_0! \equiv a_0! \pmod p \quad &\iff \quad a_0!/p^0 \equiv (-1)^0 \prod_{0\le i\le 0} a_i! \pmod p \\ &\iff \quad n!/p^\mu \equiv (-1)^\mu \prod_{0\le i\le k} a_i! \pmod p. \end{aligned}$$

Then, assuming as an inductive hypothesis for $n_k = \sum_{0\le i\le k} a_i p^i$ and $\mu_k = \sum_{1\le i\le k} \left\lfloor \frac{n_k}{p^i} \right\rfloor$ that

$$n_k!/p^{\mu_k} \equiv (-1)^{\mu_k} \prod_{0\le i\le k} a_i! \pmod p,$$

we must show for $n_{k+1} = a_{k+1}p^{k+1} + n_k$ and $\mu_{k+1} = \left\lfloor \frac{n_{k+1}}{p^{k+1}} \right\rfloor + \mu_k$ that

$$n_{k+1}!/p^{\mu_{k+1}} \equiv (-1)^{\mu_{k+1}} \prod_{0\le i\le k+1} a_i! \pmod p.$$

(Note that the equality for μ_{k+1} holds since n_{k+1} has grown by a multiple of p; namely, $a_{k+1}p^k$.)

But by Wilson's theorem, all the terms between n_k+1 and n_{k+1} that are *not* multiples of p may be collected into sets of size $p-1$ whose product is congruent to -1 modulo p, and there are precisely $\left\lfloor \frac{n_{k+1}}{p^{k+1}} \right\rfloor$ of these products, with a_{k+1} left over, congruent to $a_{k+1}!$ modulo p. That is

$$\prod_{n_k+1\le i\le n_{k+1}} i \Big/ p^{\left\lfloor \frac{n_{k+1}}{p^{k+1}} \right\rfloor} \equiv (-1)^{\left\lfloor \frac{n_{k+1}}{p^{k+1}} \right\rfloor} a_{k+1}! \pmod p.$$

Multiplying this by the inductive hypothesis yields

$$\begin{aligned} &\left(\prod_{n_k+1\le i\le n_{k+1}} i \Big/ p^{\left\lfloor \frac{n_{k+1}}{p^{k+1}} \right\rfloor} \right) n_k!/p^{\mu_k} \equiv (-1)^{\left\lfloor \frac{n_{k+1}}{p^{k+1}} \right\rfloor} a_{k+1}! (-1)^{\mu_k} \prod_{0\le i\le k} a_i! \pmod p \\ \iff \quad &n_k! \prod_{n_k+1\le i\le n_{k+1}} i \Big/ p^{\left\lfloor \frac{n_{k+1}}{p^{k+1}} \right\rfloor + \mu_k} \equiv (-1)^{\left\lfloor \frac{n_{k+1}}{p^{k+1}} \right\rfloor + \mu_k} a_{k+1}! \prod_{0\le i\le k} a_i! \pmod p \\ \iff \quad &n_{k+1}!/p^{\mu_{k+1}} \equiv (-1)^{\mu_{k+1}} \prod_{0\le i\le k+1} a_i! \pmod p. \end{aligned}$$

Therefore

$$n!/p^\mu \equiv (-1)^\mu \prod_{0\le i\le k} a_i! \pmod p$$

as we needed to show. □

15. [*HM15*] The *permanent* of a square matrix is defined by the same expansion as the determinant except that each term of the permanent is given a plus sign while the determinant alternates between plus and minus. Thus the permanent of

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

is $aei + bfg + cdh + gec + hfa + idb$. What is the permanent of

$$\begin{pmatrix} 1\times 1 & 1\times 2 & \dots & 1\times n \\ 2\times 1 & 2\times 2 & \dots & 2\times n \\ \vdots & \vdots & \ddots & \vdots \\ n\times 1 & n\times 2 & \dots & n\times n \end{pmatrix}?$$

The permanent of a square matrix may be defined recursively as

$$\operatorname{perm}([a_{ij}]_n) = \begin{cases} a_{11} & \text{if } n = 1 \\ \sum_{1\le j\le n} a_{ij} + \operatorname{perm}(\operatorname{submatrix}(a_i j)) & \text{otherwise.} \end{cases}$$

In the case that $a_{ij} = i\times j$, we may simply add

$$(1\times 1) + (2\times 2) + \cdots + (n\times n) = (n!)^2,$$

and we do this for $n!$ terms, yielding a total sum of

$$n!(n!)^2 = (n!)^3.$$

16. [*HM15*] Show that the infinite sum in Eq. (11) does not converge unless n is a nonnegative integer.

The infinite sum in Eq. (11),

$$\sum_{k\ge 0}\left(\left(\sum_{0\le j\le k}\frac{(-1)^j}{j!}\right)\left(\prod_{0\le j<k}(n-j)\right)\right),$$

does not converge unless n is a nonnegative integer, since if $n < 0$, the product $\prod_{0\le j<k}(n-j)$ never vanishes as the coefficients

$$\lim_{k\to\infty}\sum_{0\le j\le k}\frac{(-1)^j}{j!} = \frac{1}{e}.$$

(In the case that $n \ge 0$, the product eventually vanishes with a factor of zero.)

17. [*HM20*] Prove that the infinite product

$$\prod_{n\ge 1}\frac{(n+\alpha_1)\dots(n+\alpha_k)}{(n+\beta_1)\dots(n+\beta_k)}$$

equals $\Gamma(1+\beta_1)\dots\Gamma(1+\beta_k)/\Gamma(1+\alpha_1)\dots\Gamma(1+\alpha_k)$, if $\alpha_1+\cdots+\alpha_k = \beta_1+\cdots+\beta_k$ and if none of the β's is a negative integer.

Proposition. $\prod_{n\ge 1}\prod_{1\le i\le k}\frac{n+\alpha_i}{n+\beta_i} = \prod_{1\le i\le k}\frac{\Gamma(1+\beta_i)}{\Gamma(1+\alpha_i)}$ *if* $\sum_{1\le i\le k}\alpha_i = \sum_{1\le i\le k}\beta_i$ *and* $\beta_i \ge 0$ *for* $1\le i\le k$.

Proof. Let α_i and β_i be arbitrary integer sequences such that $\sum_{1\le i\le k}\alpha_i = \sum_{1\le i\le k}\beta_i$ and $\beta_i \ge 0$ for $1 \le i \le k$. We must show that

$$\prod_{n\ge 1}\prod_{1\le i\le k}\frac{n+\alpha_i}{n+\beta_i} = \prod_{1\le i\le k}\frac{\Gamma(1+\beta_i)}{\Gamma(1+\alpha_i)}.$$

But

$$\begin{aligned}
\prod_{1\le i\le k}\frac{\Gamma(1+\beta_i)}{\Gamma(1+\alpha_i)} &= \lim_{m\to\infty}\prod_{1\le i\le k}\frac{m^{1+\beta_i}m!\prod_{0\le j\le m}(1+\alpha_i+j)}{m^{1+\alpha_i}m!\prod_{0\le j\le m}(1+\beta_i+j)}\\
&= \lim_{m\to\infty}\prod_{1\le i\le k}\frac{m^{\beta_i}\prod_{0\le j\le m}(1+\alpha_i+j)}{m^{\alpha_i}\prod_{0\le j\le m}(1+\beta_i+j)}\\
&= \lim_{m\to\infty}\frac{m^{\sum_{1\le i\le k}\beta_i}}{m^{\sum_{1\le i\le k}\alpha_i}}\prod_{1\le i\le k}\frac{\prod_{0\le j\le m}(1+\alpha_i+j)}{\prod_{0\le j\le m}(1+\beta_i+j)}\\
&= \lim_{m\to\infty}\prod_{1\le i\le k}\prod_{0\le j\le m}\frac{1+\alpha_i+j}{1+\beta_i+j}\\
&= \lim_{m\to\infty}\prod_{0\le j\le m}\prod_{1\le i\le k}\frac{1+\alpha_i+j}{1+\beta_i+j}\\
&= \lim_{m\to\infty}\prod_{1\le n\le m}\prod_{1\le i\le k}\frac{n+\alpha_i}{n+\beta_i}\\
&= \prod_{n\ge 1}\prod_{1\le i\le k}\frac{n+\alpha_i}{n+\beta_i}
\end{aligned}$$

as we needed to show. □

18. [*M20*] Assume that $\pi/2 = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots$. (This is "Wallis's product," obtained by J. Wallis in 1655, and we will prove it in exercise 1.2.6-43.) Using the previous exercise, prove that $\left(\frac{1}{2}\right)! = \sqrt{\pi}/2$.

Proposition. $\left(\frac{1}{2}\right)! = \sqrt{\pi}/2$.

Proof. We must show that

$$\left(\frac{1}{2}\right)! = \sqrt{\pi}/2.$$

But according to exercise 17 with $\alpha_1 = \alpha_2 = 0$, $\beta_1 = -1/2$, and $\beta_2 = 1/2$ so that

$\sum_{1\le i\le 2}\alpha_i = \sum_{1\le i\le 2}\beta_i$ and $\beta_i \ge 0$ for $1 \le i \le 2$,

$$\begin{aligned}
\frac{\sqrt{\pi}}{2} &= \sqrt{\frac{1}{2}\frac{\pi}{2}} \\
&= \sqrt{\frac{1}{2}\prod_{n\ge 1}\frac{2n}{2n-1}\frac{2n}{2n+1}} \\
&= \sqrt{\frac{1}{2}\prod_{n\ge 1}\frac{n}{n-\frac{1}{2}}\frac{n}{n+\frac{1}{2}}} \\
&= \sqrt{\frac{1}{2}\prod_{n\ge 1}\frac{n+\alpha_1}{n+\beta_1}\frac{n+\alpha_2}{n+\beta_2}} \\
&= \sqrt{\frac{1}{2}\prod_{n\ge 1}\prod_{1\le k\le 2}\frac{n+\alpha_k}{n+\beta_k}} \\
&= \sqrt{\frac{1}{2}\prod_{1\le k\le 2}\frac{\Gamma(1+\beta_k)}{\Gamma(1+\alpha_k}} \\
&= \sqrt{\frac{1}{2}\prod_{1\le k\le 2}\frac{\Gamma(1+\beta_k)}{\Gamma(1+\alpha_k)}} \\
&= \sqrt{\frac{1}{2}\frac{\Gamma\left(1+\beta_1\right)}{\Gamma\left(1+\alpha_1\right)}\frac{\Gamma\left(1+\beta_2\right)}{\Gamma\left(1+\alpha_2\right)}} \\
&= \sqrt{\frac{1}{2}\frac{\Gamma\left(1-\frac{1}{2}\right)}{\Gamma\left(1\right)}\frac{\Gamma\left(1+\frac{1}{2}\right)}{\Gamma\left(1\right)}} \\
&= \sqrt{\frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1\right)}\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(1\right)}} \\
&= \sqrt{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)} \\
&= \sqrt{\frac{2}{2}\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)} \\
&= \sqrt{\Gamma\left(\frac{3}{2}\right)^2} \\
&= \Gamma\left(\frac{3}{2}\right) \\
&= \left(\frac{1}{2}\right)!
\end{aligned}$$

as we needed to show. □

[Wallis's own heuristic "proof" can be found in D. J. Struik's *Source Book in Mathematics* (Harvard University Press, 1969), 244–253.]

19. [*HM22*] Denote the quantity appearing after "$\lim_{m\to\infty}$" in Eq. (15) by $\Gamma_m(x)$. Show that

$$\Gamma_m(x) = \int_0^m \left(1-\frac{t}{m}\right)^m t^{x-1}dt = m^x\int_0^1 (1-t)^m t^{x-1}dt, \qquad \text{if } x>0.$$

Proposition. $\Gamma_m(x) = \int_0^m \left(1 - \frac{t}{m}\right)^m t^{x-1}dt = m^x \int_0^1 (1-t)^m t^{x-1}dt$ *if* $x > 0$.

Proof. Let x be an arbitrary positive real number so that $x > 0$. We must show that

$$\Gamma_m(x) = \int_0^m \left(1 - \frac{t}{m}\right)^m t^{x-1}dt = m^x \int_0^1 (1-t)^m t^{x-1}dt.$$

But by substituting mt for t

$$\begin{aligned}\int_0^m \left(1 - \frac{t}{m}\right)^m t^{x-1}dt &= \int_0^{t/t} \left(1 - \frac{mt}{m}\right)^m (mt)^{x-1}dmt \\ &= \int_0^1 (1-t)^m\, m^x t^{x-1}dt \\ &= m^x \int_0^1 (1-t)^m\, t^{x-1}dt.\end{aligned}$$

Then, let

$$f_m(x) = \int_0^1 (1-t)^m\, t^{x-1}dt.$$

We may prove by induction on m that

$$f_m(x) = \frac{m!}{\prod_{0 \le k \le m}(x+k)}.$$

If $m = 0$, clearly

$$\begin{aligned}f_0(x) &= \int_0^1 (1-t)^0\, t^{x-1}dt \\ &= \int_0^1 t^{x-1}dt \\ &= \frac{t^x}{x}\bigg|_0^1 \\ &= \frac{1^x}{x} - \frac{0^x}{x} \\ &= \frac{1-0}{x} \\ &= \frac{1}{x} \\ &= \frac{0!}{\prod_{0 \le k \le 0}(x+k)}.\end{aligned}$$

Then, assuming

$$f_m(x) = \frac{m!}{\prod_{0 \le k \le m}(x+k)},$$

we must show that

$$f_{m+1}(x) = \frac{(m+1)!}{\prod_{0 \le k \le m+1}(x+k)}.$$

But by integration by parts, since $\frac{d}{dx}\frac{t^x}{x} = t^{x-1}$ and $\frac{d}{dt}(1-t)^{m+1} = -(m+1)(1-t)^m$,

$$\begin{aligned}
f_{m+1}(x) &= \int_0^1 (1-t)^{m+1} t^{x-1} dt \\
&= (1-t)^{m+1}\frac{t^x}{x}\Big|_0^1 - \int_0^1 -(m+1)(1-t)^m \frac{t^x}{x} dt \\
&= (1-1)^{m+1}\frac{1^x}{x} - (1-0)^{m+1}\frac{0^x}{x} + \frac{m+1}{x}\int_0^1 (1-t)^m t^x dt \\
&= 0 - 0 + \frac{m+1}{x}\int_0^1 (1-t)^m t^x dt \\
&= \frac{m+1}{x} f_m(x+1).
\end{aligned}$$

And so, by the inductive hypothesis,

$$\begin{aligned}
f_{m+1}(x) &= \frac{m+1}{x} f_m(x+1) \\
&= \frac{m+1}{x}\frac{m!}{\prod_{0\le k\le m}(x+1+k)} \\
&= \frac{(m+1)m!}{x\prod_{1\le k\le m+1}(x+k)} \\
&= \frac{(m+1)!}{\prod_{0\le k\le m+1}(x+k)},
\end{aligned}$$

so that

$$f_m(x) = \frac{m!}{\prod_{0\le k\le m}(x+k)}.$$

Finally

$$\begin{aligned}
\int_0^m \left(1-\frac{t}{m}\right)^m t^{x-1} dt &= m^x \int_0^1 (1-t)^m t^{x-1} dt \\
&= m^x f_m(x) \\
&= m^x \frac{m!}{\prod_{0\le k\le m}(x+k)} \\
&= \frac{m^x m!}{\prod_{0\le k\le m}(x+k)} \\
&= \Gamma_m(x)
\end{aligned}$$

as we needed to show. □

20. [*HM21*] Using the fact that $0 \le e^{-t} - (1-t/m)^m \le t^2 e^{-t}/m$, if $0 \le t \le m$, and the previous exercise, show that $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$, if $x > 0$.

Proposition. *$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$, if $x > 0$.*

Proof. Let x be an arbitrary positive real number such that $x > 0$. We must show that

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

Let

$$g(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

It is sufficient to show that $g(x) - \Gamma(x) = 0$.

Note that if $0 \le t \le m$,

$$\begin{aligned} 1 + x \le e^x \quad &\iff \quad 1 \pm t/m \le e^{\pm t/m} \\ &\iff \quad (1 \pm t/m)^m \le e^{\pm t} \end{aligned}$$

and from exercise 1.2.1-9,

$$\begin{aligned} e^{-t} &\ge (1 - t/m)^m \\ &= e^{-t}(1 - t/m)^m e^t \\ &\ge e^{-t}(1 - t/m)^m (1 + t/m)^m \\ &= e^{-t}(1 - t^2/m^2)^m \\ &\ge e^{-t}(1 - t^2/m) \end{aligned}$$

so that

$$0 \le e^{-t} - (1 - t/m)^m \le t^2 e^{-t}/m.$$

Then since $x + 1 \ge 2$,

$$\begin{aligned} &0 \le e^{-t} - (1 - t/m)^m \le t^{x+1} e^{-t}/m \\ \iff \quad & 0 \le \int_0^m e^{-t} - (1 - t/m)^m dt \le \frac{1}{m} \int_0^m t^{x+1} e^{-t} dt < \frac{1}{m} \int_0^\infty t^{x+1} e^{-t} dt \\ \iff \quad & 0 \le \int_0^\infty e^{-t} - (1 - t/m)^m dt \le \frac{1}{m} \int_0^\infty t^{x+1} e^{-t} dt \\ \iff \quad & 0 \le \int_0^\infty e^{-t} - (1 - t/m)^m dt \le 0 \\ \iff \quad & \int_0^\infty e^{-t} - (1 - t/m)^m dt = 0 \\ \iff \quad & \int_0^\infty e^{-t} t^{x-1} dt - \int_0^\infty (1 - t/m)^m t^{x-1} dt = 0 \\ \iff \quad & g(x) - \Gamma(x) = 0 \end{aligned}$$

as we needed to show. □

21. [*HM25*] (L. F. A. Arbogast, 1800.) Let $D_x^k u$ represent the kth derivative of a function u with respect to x. The chain rule states that $D_x^1 w = D_u^1 w D_x^1 u$. If we apply this to second derivatives, we find $D_x^2 w = D_u^2 w (D_x^1 u)^2 + D_u^1 w D_x^2 u$. Show that the *general formula* is

$$D_x^n w = \sum_{j=0}^{n} \sum_{\substack{k_1 + k_2 + \cdots + k_n = j \\ k_1 + 2k_2 + \cdots + nk_n = n \\ k_1, k_2, \ldots, k_n \ge 0}} D_u^j w \frac{n!}{k_1!(1!)^{k_1} \ldots k_n!(n!)^{k_n}} (D_x^1 u)^{k_1} \ldots (D_x^n u)^{k_n}.$$

Proposition. $D_x^n f = \sum_{0 \le j \le n} \sum_{\substack{k_1 + k_2 + \cdots + k_n = j \\ k_1 + 2k_2 + \cdots + nk_n = n \\ k_1, k_2, \ldots, k_n \ge 0}} n! D_g^j f \prod_{1 \le i \le n} \frac{(D_x^i g)^{k_i}}{k_i!(i!)^{k_i}}$.

Proof. Let f be an arbitrary function of g, g an arbitarary function of x, and n a positive integer. We must show that that the nth derivative of f with respect to x is

$$D_x^n f = \sum_{0 \le j \le n} \sum_{\substack{k_1 + k_2 + \cdots + k_n = j \\ k_1 + 2k_2 + \cdots + nk_n = n \\ k_1, k_2, \ldots, k_n \ge 0}} n! D_g^j f \prod_{1 \le i \le n} \frac{(D_x^i g)^{k_i}}{k_i!(i!)^{k_i}}.$$

As done by T. A.[1], it suffices to analyze the coefficients for any function f. Let $f = e^{pg(x)}$. By Taylor's theorem for an arbitrary h,

$$e^{pg(x+h)} = \sum_{n\geq 0} \frac{h^n}{n!} D_x^n f.$$

But also, by expanding $g(x+h)$ and developing the product,

$$\begin{aligned} e^{pg(x+h)} &= e^{pg(x)} \prod_{n\geq 1} e^{pD_x^n g \frac{h^n}{n!}} \\ &= \sum_{n\geq 0} h^n \sum_{0\leq j\leq n} p^j e^{pg(x)} \sum_{\substack{k_1+k_2+\cdots+k_n=j \\ k_1+2k_2+\cdots+nk_n=n \\ k_1,k_2,\ldots,k_n\geq 0}} \prod_{1\leq i\leq n} \frac{(D_x^i g)^{k_i}}{k_i!(i!)^{k_i}} \\ &= \sum_{n\geq 0} h^n \sum_{0\leq j\leq n} D_g^j f \sum_{\substack{k_1+k_2+\cdots+k_n=j \\ k_1+2k_2+\cdots+nk_n=n \\ k_1,k_2,\ldots,k_n\geq 0}} \prod_{1\leq i\leq n} \frac{(D_x^i g)^{k_i}}{k_i!(i!)^{k_i}}. \end{aligned}$$

Equating these two yields

$$\begin{aligned} \sum_{n\geq 0} \frac{h^n}{n!} D_x^n f &= \sum_{n\geq 0} h^n \sum_{0\leq j\leq n} D_g^j f \sum_{\substack{k_1+k_2+\cdots+k_n=j \\ k_1+2k_2+\cdots+nk_n=n \\ k_1,k_2,\ldots,k_n\geq 0}} \prod_{1\leq i\leq n} \frac{(D_x^i g)^{k_i}}{k_i!(i!)^{k_i}} \\ \iff \quad \frac{1}{n!} D_x^n f &= \sum_{0\leq j\leq n} D_g^j f \sum_{\substack{k_1+k_2+\cdots+k_n=j \\ k_1+2k_2+\cdots+nk_n=n \\ k_1,k_2,\ldots,k_n\geq 0}} \prod_{1\leq i\leq n} \frac{(D_x^i g)^{k_i}}{k_i!(i!)^{k_i}} \\ \iff \quad D_x^n f &= \sum_{0\leq j\leq n} \sum_{\substack{k_1+k_2+\cdots+k_n=j \\ k_1+2k_2+\cdots+nk_n=n \\ k_1,k_2,\ldots,k_n\geq 0}} n! D_g^j f \prod_{1\leq i\leq n} \frac{(D_x^i g)^{k_i}}{k_i!(i!)^{k_i}} \end{aligned}$$

and hence the result. □

[*Bull. Amer. Math. Soc.* **44** (1938), 395–398]

[*Du Calcul des Dérivations* (Strasbourg: 1800), §52]

[*Quarterly J. Math.* **1** (1857), 359-360]

. . . see the paper by I. J. Good, *Annals of Mathematical Statistics* **32** (1961), 540–541.

▶ **22.** [*HM20*] Try to put yourself in Euler's place, looking for a way to generalize $n!$ to noninteger values of n. Since $(n+\frac{1}{2})!/n!$ times $((n+\frac{1}{2})+\frac{1}{2})!/(n+\frac{1}{2})!$ equals $(n+1)!/n! = n+1$, it seems natural that $(n+\frac{1}{2})!/n!$ should be approximately $\sqrt(n)$. Similarly, $(n+\frac{1}{3})!/n!$ should be $\approx \sqrt[3]{n}$. Invent a hypothesis about the ratio $(n+x)!/n!$ as n approaches infinity. Is your hypothesis correct when x is an integer? Does it tell anything about the appropriate value of $x!$ when x is not an integer?

[1] T. A. [J. F. C. Tiburce Abadie], Sur la différentiation des fonctions de fonctions, *Nouvelles Annales de Mathématiques* **9** (1850) 119-125.

Observing that

$$\frac{(n+\frac{1}{2})!}{n!} \approx \sqrt{n} \text{ and}$$
$$\frac{(n+\frac{1}{3})!}{n!} \approx \sqrt[3]{n},$$

we might hypothesize that

$$\lim_{n\to\infty} \frac{(n+x)!}{n!n^x} = 1.$$

When x is an integer, the equality holds, as

$$\begin{aligned}
\lim_{n\to\infty} \frac{(n+x)!}{n!n^x} &= \lim_{n\to\infty} \frac{\prod_{1\le k\le x}(n+k)}{n^x} \\
&= \lim_{n\to\infty} \frac{n^x \prod_{1\le k\le x}\left(1+\frac{k}{n}\right)}{n^x} \\
&= \lim_{n\to\infty} \prod_{1\le k\le x} \left(1+\frac{k}{n}\right) \\
&= 1.
\end{aligned}$$

It tells us something about the appropriate value of $x!$ when x is not an integer as well, since

$$\begin{aligned}
1 = \lim_{n\to\infty} \frac{(n+x)!}{n!n^x} &= x! \lim_{n\to\infty} \frac{\prod_{1\le k\le n}(x+k)}{n!n^x} \\
\iff \quad x! &= \lim_{n\to\infty} \frac{n^x n!}{\prod_{1\le k\le n}(x+k)} = \Gamma(x+1).
\end{aligned}$$

23. [*HM20*] Prove (16), given that $\pi z \prod_{n=1}^{\infty}(1 - z^2/n^2) = \sin \pi z$.

Proposition. $(-z)!\Gamma(z) = \frac{\pi}{\sin \pi z}$ *for z not an integer.*

Proof. Let z be an arbitrary real number, not an integer. We must show that

$$(-z)!\Gamma(z) = \frac{\pi}{\sin \pi z}.$$

But given

$$\begin{aligned}
\pi z \prod_{m\ge1} \left(1 - \frac{z^2}{m^2}\right) &= \sin \pi z \\
\iff \quad \frac{\pi z}{\sin \pi z} &= \frac{1}{\prod_{m\ge1}\left(1-\frac{z^2}{m^2}\right)}
\end{aligned}$$

we have

$$\begin{aligned}
z(-z)!\Gamma(z) &= z \lim_{m\to\infty} \frac{m^{-z}m!}{\prod_{1\le k\le m}(-z+k)} \lim_{m\to\infty} \frac{m^{z}m!}{z\prod_{1\le k\le m}(z+k)} \\
&= \lim_{m\to\infty} \frac{m^{-z}m!m^{z}m!}{\prod_{1\le k\le m}(-z+k)(z+k)} \\
&= \lim_{m\to\infty} \frac{(m!)^2}{\prod_{1\le k\le m} k^2\left(1-\frac{z}{k}\right)\left(1+\frac{z}{k}\right)} \\
&= \lim_{m\to\infty} \frac{(m!)^2}{(m!)^2\prod_{1\le k\le m} \left(1-\frac{z}{k}\right)\left(1+\frac{z}{k}\right)} \\
&= \lim_{m\to\infty} \frac{1}{\prod_{1\le k\le m} \left(1-\frac{z^2}{k^2}\right)} \\
&= \frac{1}{\prod_{m\ge 1} \left(1-\frac{z^2}{k^2}\right)} \\
&= \frac{\pi z}{\sin \pi z}.
\end{aligned}$$

Finally, dividing both sides by z yields

$$(-z)!\Gamma(z) = \frac{\pi}{\sin \pi z}$$

as we needed to show. □

▶ **24.** [*HM21*] Prove the handy inequalities

$$\frac{n^n}{e^{n-1}} \le n! \le \frac{n^{n+1}}{e^{n-1}}, \qquad \text{integer } n \ge 1.$$

[*Hint:* $1 + x \le e^x$ for all real x; hence $(k+1)/k \le e^{1/k} \le k/(k-1)$.]

Proposition. $\frac{n^n}{e^{n-1}} \le n! \le \frac{n^{n+1}}{e^{n-1}}$ *for integer* $n \ge 1$.

Proof. Let n be an arbitrary integer such that $n \ge 1$. We must show that

$$\frac{n^n}{e^{n-1}} \le n! \le \frac{n^{n+1}}{e^{n-1}}.$$

Note that since $1 + x \le e^x$ for all real x,

$$\begin{aligned}
1 + \frac{1}{k} \le e^{\frac{1}{k}} \quad &\iff \quad \frac{k+1}{k} \le e^{\frac{1}{k}} \\
&\iff \quad \frac{(k+1)^k}{k^k} \le e,
\end{aligned}$$

and

$$\begin{aligned}
1 - \frac{1}{k+1} \le e^{-\frac{1}{k+1}} \quad &\iff \quad \frac{k}{k+1} \le e^{-\frac{1}{k+1}} \\
&\iff \quad \frac{k^{k+1}}{(k+1)^{k+1}} \le e^{-1} \\
&\iff \quad \frac{(k+1)^{k+1}}{k^{k+1}} \ge e.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{n^n}{n!} &= \frac{\prod_{1\le k\le n} n}{\prod_{1\le k\le n} k} \\
&= \frac{\prod_{1\le k\le n} nk^{k-1}}{\prod_{1\le k\le n} kk^{k-1}} \\
&= \frac{\prod_{1\le k\le n} nk^{k-1}}{\prod_{1\le k\le n} k^k} \\
&= \frac{n^n}{n^n}\frac{\prod_{1\le k\le n} k^{k-1}}{\prod_{1\le k\le n-1} k^k} \\
&= \frac{\prod_{2\le k\le n} k^{k-1}}{\prod_{1\le k\le n-1} k^k} \\
&= \frac{\prod_{1\le k\le n-1}(k+1)^k}{\prod_{1\le k\le n-1} k^k} \\
&= \prod_{1\le k\le n-1} \frac{(k+1)^k}{k^k} \\
&\le \prod_{1\le k\le n-1} e \\
&= e^{n-1},
\end{aligned}$$

and so

$$\frac{n^n}{e^{n-1}} \le n!.$$

Then also

$$\begin{aligned}
\frac{n^{n+1}}{n!} &= \frac{\prod_{1\le k\le n+1} n}{\prod_{1\le k\le n} k} \\
&= \frac{\prod_{1\le k\le n+1} nk^k}{\prod_{1\le k\le n} kk^k} \\
&= \frac{\prod_{1\le k\le n+1} nk^k}{\prod_{1\le k\le n} k^{k+1}} \\
&= \frac{n^{n+1}}{n^{n+1}}\frac{\prod_{1\le k\le n} k^k}{\prod_{1\le k\le n-1} k^{k+1}} \\
&= \frac{\prod_{2\le k\le n} k^k}{\prod_{1\le k\le n-1} k^{k+1}} \\
&= \frac{\prod_{1\le k\le n-1}(k+1)^{k+1}}{\prod_{1\le k\le n-1} k^{k+1}} \\
&= \prod_{1\le k\le n-1} \frac{(k+1)^{k+1}}{k^{k+1}} \\
&\ge \prod_{1\le k\le n-1} e \\
&= e^{n-1},
\end{aligned}$$

and so

$$n! \le \frac{n^{n+1}}{e^{n-1}}.$$

Therefore,

$$\frac{n^n}{e^{n-1}} \leq n! \leq \frac{n^{n+1}}{e^{n-1}}$$

as we needed to show. □

25. [*M20*] Do factorial powers satisfy a law analogous to the ordinary law of exponents, $x^{m+n} = x^m x^n$?

Factorial powers satisfy laws analogous to the ordinary law of exponents. In particular,

$$x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}$$
$$x^{\overline{m+n}} = x^{\overline{m}}(x+m)^{\overline{n}},$$

since

$$\begin{aligned} x^{\underline{m+n}} &= \frac{x!}{(x-(m+n))!} \\ &= \frac{x!}{(x-m-n)!} \\ &= \frac{x!}{(x-m-n)!}\frac{(x-m)!}{(x-m)!} \\ &= \frac{x!}{(x-m)!}\frac{(x-m)!}{(x-m-n)!} \\ &= x^{\underline{m}}(x-m)^{\underline{n}} \end{aligned}$$

and

$$\begin{aligned} x^{\overline{m+n}} &= = \frac{\Gamma(x+m+n)}{\Gamma(x)} \\ &= \frac{\Gamma(x+m+n)}{\Gamma(x)}\frac{\Gamma(x+m)}{\Gamma(x+m)} \\ &= \frac{\Gamma(x+m)}{\Gamma(x)}\frac{\Gamma(x+m+n)}{\Gamma(x+m)} \\ &= x^{\overline{m}}(x+m)^{\overline{n}}. \end{aligned}$$