# Exercises from Section 1.2.8

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**1.** [*10*] What is the answer to Leonardo Fibonacci's original problem: How many pairs of rabbits are present after a year?

The original problem assumes we start with a single pair of rabbits and a monthly period. Let $k$ represent the number of months that have passed and our sequence by $S_k$, so that that initially we start with

$$S_0 = 1 = F_2 = F_{0+2} \text{ pair.}$$

After a year, we have

$$S_{12} = F_{12+2} = F_{14} = 377 \text{ pairs,}$$

and in general, after $k$ months, we have

$$S_k = F_{k+2} \text{ pairs.}$$

▶ **2.** [*20*] In view of Eq. (15), what is the approximate value of $F_{1000}$? (Use logarithms found in Appendix A.)

Given Eq. (15)

$$F_n = \phi^n/\sqrt{5} \text{ rounded to the nearest integer,}$$

we may use the logarithms found in Appendix A to find that

$$\begin{aligned} F_{1000} &\approx e^{\ln(\phi^{1000}/\sqrt{5})} \\ &= e^{1000\ln\phi - \frac{1}{2}\ln 5} \\ &= e^{1000\ln\phi - \frac{1}{2}\ln 5} \\ &= e^{1000\ln\phi - \frac{1}{2}(\ln 10 - \ln 2)} \\ &\approx e^{408.40711} \\ &\approx 10^{408.40711/\ln 10} \\ &\approx 10^{208.63816} \\ &\approx 4.34666 \times 10^{208}.. \end{aligned}$$

That is, $F_{1000}$ is a 209-digit number whose leading digit is 4.

**3.** [*25*] Write a computer program that calculates and prints $F_1$ through $F_{1000}$ in decimal notation. (The previous exercise determines the size of numbers that must be handled.)

The following Java code calculates and prints $F_1$ through $F_{1000}$, by assuming nonnegative integers no larger than 209 digits.

```
class FibonacciNumber {

   public FibonacciNumber(int initialValue) {
      decimalDigits = new int[209];
      for (decimalDigitCount = 0; initialValue != 0; ++decimalDigitCount) {
         decimalDigits[decimalDigitCount] = initialValue % 10;
         initialValue /= 10;
      }
   }
```

```
    public FibonacciNumber plus(FibonacciNumber fibonacciNumber) {
        FibonacciNumber sum = new FibonacciNumber(0);
        int carry = 0;
        for (
            int k = 0;
            k < Math.max(decimalDigitCount, fibonacciNumber.decimalDigitCount);
            ++k
        ) {
            int thisDigit = (k < decimalDigitCount) ?
                decimalDigits[k] : 0;
            int thatDigit = (k < fibonacciNumber.decimalDigitCount) ?
                fibonacciNumber.decimalDigits[k] : 0;
            int digitSum = thisDigit + thatDigit + carry;
            sum.decimalDigits[sum.decimalDigitCount++] = digitSum % 10;
            carry = digitSum / 10;
        }
        if (carry > 0) {
            sum.decimalDigits[sum.decimalDigitCount++] = carry;
        }
        return (sum);
    }

    public String toString() {
        StringBuilder stringBuilder = new StringBuilder();
        for (int k = decimalDigitCount - 1; k >= 0; --k) {
            stringBuilder.append(decimalDigits[k]);
        }
        if (stringBuilder.length() == 0) {
            stringBuilder.append(0);
        }
        return (stringBuilder.toString());
    }

    private int[] decimalDigits;
    private int decimalDigitCount;

}

FibonacciNumber[] fibonacci = new FibonacciNumber[1000];
int k = 0;
System.out.println(fibonacci[k] = new FibonacciNumber(1));
++k;
System.out.println(fibonacci[k] = fibonacci[k - 1]);
for (++k; k < fibonacci.length; ++k) {
    System.out.println(fibonacci[k] = fibonacci[k - 1].plus(fibonacci[k - 2]));
}
```

The first thirty numbers generated are listed below,

| $n$ | $F_n$ |
|---|---|
| 1 | 1 |
| 2 | 1 |
| 3 | 2 |
| 4 | 3 |
| 5 | 5 |
| 6 | 8 |
| 7 | 13 |
| 8 | 21 |
| 9 | 34 |
| 10 | 55 |
| 11 | 89 |
| 12 | 144 |
| 13 | 233 |
| 14 | 377 |
| 15 | 610 |
| 16 | 987 |
| 17 | 1597 |
| 18 | 2584 |
| 19 | 4181 |
| 20 | 6765 |
| 21 | 10946 |
| 22 | 17711 |
| 23 | 28657 |
| 24 | 46368 |
| 25 | 75025 |
| 26 | 121393 |
| 27 | 196418 |
| 28 | 317811 |
| 29 | 514229 |
| 30 | 832040 |

and $F_{1000}$ is printed as anticipated, a 209-digit number whose leading digit is 4:

```
43466 55768 69374 56435 68852 76750 40625 80256 46605 17371
78040 24817 29089 53655 54179 49051 89040 38798 40079 25516
92959 22593 08032 26347 75209 68962 32398 73322 47116 16429
96440 90653 31879 38298 96964 99285 16003 70447 61377 95166
84922 8875.
```

▶ **4.** [*14*] Find all $n$ for which $F_n = n$.

Manually inspecting $F_n$ until $F_{n-1} > n$

| $n$ | $F_n$ |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 1 |
| 3 | 2 |
| 4 | 3 |
| 5 | 5 |
| 6 | 8 |
| 7 | 13 |

reveals $F_n = n$ for $n = 0$, 1, and 5. For $n > 5$, $F_n$ increases faster than $n$, letting us conclude that these are the only $n$, as may be seen by the inductive argument that follows.

In the case that $n = 6$, clearly $F_n = F_6 = 8 > 6 = n$. Similarly, in the case that $n = 7$, $F_n = F_7 = 13 > 7 = n$. Then, assuming $F_n > n$ for $n > 5$, we must show that $F_{n+1} > n+1$. But

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ &> n + n - 1 \\ &> n + 1 \end{aligned}$$

since $n > 5$ by hypothesis, and hence the conclusion.

**5.** [*20*] Find all $n$ for which $F_n = n^2$.

Manually inspecting $F_n$ until $F_{n-1} > n^2$

| $n$ | $n^2$ | $F_n$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |
| 2 | 4 | 1 |
| 3 | 9 | 2 |
| 4 | 16 | 3 |
| 5 | 25 | 5 |
| 6 | 36 | 8 |
| 7 | 49 | 13 |
| 8 | 64 | 21 |
| 9 | 81 | 34 |
| 10 | 100 | 55 |
| 11 | 121 | 89 |
| 12 | 144 | 144 |
| 13 | 169 | 233 |
| 14 | 196 | 377 |

reveals $F_n = n^2$ for $n = 0$, 1, and 12. For $n > 12$, $F_n$ increases faster than $n$, letting us conclude that these are the only $n$, as may be seen by the inductive argument that follows.

In the case that $n = 13$, clearly $F_n = F_{13} = 233 > 169 = 13^2 = n^2$. Similarly, in the case that $n = 14$, $F_n = F_{14} = 377 > 196 = 14^2 = n^2$. Then, assuming $F_n > n^2$ for $n > 12$, we must show that $F_{n+1} > (n+1)^2$. But

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ &> n^2 + (n-1)^2 \\ &= n^2 + n^2 - 2n + 1 \\ &> n^2 + 2n + 1 \\ &= (n+1)^2 \end{aligned}$$

since $n > 12$ by hypothesis, and hence the conclusion.

**6.** [*HM10*] Prove Eq. (5).

**Proposition.** $\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n.$

*Proof.* Let $n$ be an arbitrary positive integer. We must show that

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n.$$

In the case that $n = 1$,

$$\begin{aligned}\begin{pmatrix} F_{1+1} & F_1 \\ F_1 & F_{1-1} \end{pmatrix} &= \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^1;\end{aligned}$$

and in the case that $n = 2$,

$$\begin{aligned}\begin{pmatrix} F_{2+1} & F_2 \\ F_2 & F_{2-1} \end{pmatrix} &= \begin{pmatrix} F_3 & F_2 \\ F_2 & F_1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1+1 & 1+0 \\ 1+0 & 1+0 \end{pmatrix} \\ &= \begin{pmatrix} 1\cdot 1+1\cdot 1 & 1\cdot 1+1\cdot 0 \\ 1\cdot 1+0\cdot 1 & 1\cdot 1+0\cdot 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2.\end{aligned}$$

Then, assuming

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n,$$

we must show that

$$\begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n+1}.$$

But

$$\begin{aligned}\begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix} &= \begin{pmatrix} F_{n+1}+F_n & F_n+F_{n-1} \\ F_{n+1} & F_n \end{pmatrix} \\ &= \begin{pmatrix} 1\cdot F_{n+1}+1\cdot F_n & 1\cdot F_n+1\cdot F_{n-1} \\ 1\cdot F_{n+1}+0\cdot F_n & 1\cdot F_n+0\cdot F_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n+1}.\end{aligned}$$

as we needed to show. □

▶ **7.** [*15*] If $n$ is not a prime number, $F_n$ is not a prime number (with one exception). Prove this and find the exception.

**Proposition.** *If $n$ is not a prime number, $F_n$ is not a prime number, with the one exception being $n = 4$ where $F_4 = 3$.*

*Proof.* Let $n$ be an arbitrary nonnegative integer. We must show that if $n$ is not a prime number, $F_n$ is not a prime number, with the one exception being $n = 4$ where $F_4 = 3$.

In the case that $n = 0$ not prime, $F_0 = 0$ not prime; similarly for $n = 1$, $F_1 = 1$. Otherwise, let us assume $n > 2$ not prime, such that $d$ is a proper divisor of $n$ ($d|n$, $1 < d < n$) such that $n = dm$ for some positive integer $m$. Deduced from Eq. (6) we know that $F_d$ divides $F_n$. Since $d > 1$, $F_d \geq 1$; and since $n > 2$, $F_d < F_n$. That is, $1 \leq F_d < F_n$.

Hence, $F_n$ is not prime in all cases except where $F_d = 1$, or equivalently since $d > 1$, where $d = 2$. The only composite number $n$ that has no proper factor greater than 2 is $n = 4$, being the one exception, where $F_4 = 3$, as we needed to show. □

**8.** [*15*] In many cases it is convenient to define $F_n$ for *negative* $n$, by assuming that $F_{n+2} = F_{n+1} + F_n$ for *all* integers $n$. Explore this possibility: What is $F_{-1}$? What is $F_{-2}$? Can $F_{-n}$ be expressed in a simple way in terms of $F_n$?

Allowing $n$ to range over all integers, we require

$$F_1 = F_0 + F_{-1},$$

or equivalently,

$$\begin{aligned} F_{-1} &= F_1 - F_0 \\ &= 1 - 0 \\ &= 1. \end{aligned}$$

Similarly,

$$\begin{aligned} F_{-2} &= F_0 - F_{-1} \\ &= 0 - 1 \\ &= -1, \end{aligned}$$

and in general for nonegative $n$,

$$F_{-n} = (-1)^{n+1} F_n,$$

as is shown below.

**Proposition.** $F_{-n} = F_{-n+2} - F_{-n+1} = (-1)^{n+1} F_n$.

*Proof.* Let $n$ be an arbitrary nonnegative integer. We must show that

$$F_{-n} = F_{-n+2} - F_{-n+1} = (-1)^{n+1} F_n.$$

In the case that $n = 0$,

$$\begin{aligned} F_0 &= F_2 - F_1 \\ &= 1 - 1 \\ &= 0 \\ &= (-1)^{0+1} F_0; \end{aligned}$$

and in the case that $n = 1$,

$$\begin{aligned} F_{-1} &= F_1 - F_0 \\ &= 1 - 0 \\ &= 1 \\ &= (-1)^{1+1} F_1. \end{aligned}$$

Then, assuming

$$F_{-n} = F_{-n+2} - F_{-n+1} = (-1)^{n+1} F_n,$$

we must show that

$$F_{-(n+1)} = F_{-(n+1)+2} - F_{-(n+1)+1} = (-1)^{(n+1)+1} F_{n+1}.$$

But

$$\begin{aligned} F_{-(n+1)} &= F_{-(n+1)+2} - F_{-(n+1)+1} \\ &= F_{-n+1} - F_{-n} \\ &= (-1)^{(n-1)+1} F_{n-1} - (-1)^{n+1} F_n \\ &= (-1)^n F_{n-1} - (-1)^{n+1} F_n \\ &= (-1)^n F_{n-1} + (-1)^n F_n \\ &= (-1)^n (F_{n-1} + F_n) \\ &= (-1)^n F_{n+1} \\ &= (-1)^{n+2} F_{n+1} \\ &= (-1)^{(n+1)+1} F_{n+1} \end{aligned}$$

as we needed to show. □

**9.** [*M20*] Using the conventions of exercise 8, determine whether Eqs. (4), (6), (14), and (15) still hold when the subscripts are allowed to be *any* integers.

We determine that Eqs. (4), (6), and (14) hold if $n$ is allowed to range over all the integers, but not Eq. (15), as given by counterexample.

**Proposition.** $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ *for negative* $n$.

*Proof.* Let $n$ be an arbitrary negative integer. We must show that

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

In the case that $n = -1$,

$$F_0 F_{-2} - F_{-1}^2 = -1 = (-1)^{-1};$$

and in the case that $n = -2$,

$$F_{-1}F_{-3} - F_{-2}^2 = 2 - 1 = 1 = (-1)^{-2}.$$

Then, assuming

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n,$$

we must show that

$$F_n F_{n-2} - F_{n-1}^2 = (-1)^{n-1}.$$

But

$$\begin{aligned}
F_nF_{n-2} - F_{n-1}^2 &= (F_{n+1} - F_{n-1})(F_n - F_{n-1}) - F_{n-1}^2 \\
&= F_{n+1}F_n - F_{n-1}F_n - F_{n+1}F_{n-1} + F_{n-1}^2 - F_{n-1}^2 \\
&= F_{n+1}F_n - F_{n-1}F_n - F_{n+1}F_{n-1} \\
&= F_n(F_{n+1} - F_{n-1}) - F_{n+1}F_{n-1} \\
&= F_nF_n - F_{n+1}F_{n-1} \\
&= F_n^2 - F_{n+1}F_{n-1} \\
&= (-1)(F_{n+1}F_{n-1} - F_n^2) \\
&= (-1)(-1)^n \\
&= (-1)^{n-1}
\end{aligned}$$

as we needed to show. □

**Proposition.** $F_{n+m} = F_mF_{n+1} + F_{m-1}F_n$ *for negative* $n$.

*Proof.* Let $n$ and $m$ be arbitrary integers such that $n$ is negative and $m$ is nonnegative. We must show that

$$F_{n+m} = F_mF_{n+1} + F_{m-1}F_n.$$

In the case that $n = -1$,

$$F_{-1+m} = F_{m-1} = F_mF_0 + F_{m-1}F_{-1};$$

and in the case that $n = -2$,

$$F_{-2+m} = F_m - F_{m-1} = F_mF_{-1} + F_{m-1}F_{-2}.$$

Then, assuming

$$F_{n+m} = F_mF_{n+1} + F_{m-1}F_n$$

we must show that

$$F_{n+m-1} = F_mF_n + F_{m-1}F_{n-1}.$$

But

$$\begin{aligned}
F_{n+m-1} &= F_{n+m+1} - F_{n+m} \\
&= F_mF_{n+2} + F_{m-1}F_{n+1} - F_mF_{n+1} - F_{m-1}F_n \\
&= F_m(F_{n+2} - F_{n+1}) + F_{m-1}(F_{n+1} - F_n) \\
&= F_mF_n + F_{m-1}F_{n-1}
\end{aligned}$$

as we needed to show. □

**Proposition.** $F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \hat{\phi}^n\right)$ *for negative* $n$.

*Proof.* Let $n$ be an arbitrary negative integer. We must show that

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \hat{\phi}^n\right).$$

In the case that $n = -1$,

$$
\begin{aligned}
F_{-1} &= 1 \\
&= \frac{\sqrt{5}}{\sqrt{5}} \\
&= \frac{1}{\sqrt{5}}\left(\frac{-4\sqrt{5}}{-4}\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{2(1-\sqrt{5}) - 2(1+\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})}\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{2}{1+\sqrt{5}} - \frac{2}{1-\sqrt{5}}\right) \\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{-1}\right) \\
&= \frac{1}{\sqrt{5}}\left(\phi^{-1} - \hat{\phi}^{-1}\right);
\end{aligned}
$$

and in the case that $n = -2$,

$$
\begin{aligned}
F_{-2} &= -1 \\
&= \frac{-\sqrt{5}}{\sqrt{5}} \\
&= \frac{1}{\sqrt{5}}\left(\frac{-16\sqrt{5})}{16}\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{4(6-2\sqrt{5}) - 4(6+2\sqrt{5})}{6^2 - 4\sqrt{5}^2}\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{4}{6+2\sqrt{5}} - \frac{4}{6-2\sqrt{5}}\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{4}{(1+\sqrt{5})^2} - \frac{4}{(1-\sqrt{5})^2}\right) \\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{-2} - \left(\frac{1-\sqrt{5}}{2}\right)^{-2}\right) \\
&= \frac{1}{\sqrt{5}}\left(\phi^{-2} - \hat{\phi}^{-2}\right).
\end{aligned}
$$

Then, assuming

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \hat{\phi}^n\right),$$

we must show that

$$F_{n-1} = \frac{1}{\sqrt{5}}\left(\phi^{n-1} - \hat{\phi}^{n-1}\right).$$

But

$$
\begin{aligned}
F_{n-1} &= F_{n+1} - F_n \\
&= \frac{1}{\sqrt{5}}\left(\phi^{n+1} - \hat{\phi}^{n+1}\right) - \frac{1}{\sqrt{5}}\left(\phi^n - \hat{\phi}^n\right) \\
&= \frac{1}{\sqrt{5}}\left(\phi^{n+1} - \hat{\phi}^{n+1} - \phi^n + \hat{\phi}^n\right) \\
&= \frac{1}{\sqrt{5}}\left(\phi^{n+1} - \phi^n - \hat{\phi}^{n+1} + \hat{\phi}^n\right) \\
&= \frac{1}{\sqrt{5}}\left((\phi - 1)\,\phi^n - \left(\hat{\phi} - 1\right)\hat{\phi}^n\right) \\
&= \frac{1}{\sqrt{5}}\left(\left(\phi^2 - \phi\right)\phi^{n-1} - \left(\hat{\phi}^2 - \hat{\phi}\right)\hat{\phi}^{n-1}\right) \\
&= \frac{1}{\sqrt{5}}\left(\phi\left(\frac{1+\sqrt{5}}{2} - 1\right)\phi^{n-1} - \left(\hat{\phi}^2 - \hat{\phi}\right)\hat{\phi}^{n-1}\right) \\
&= \frac{1}{\sqrt{5}}\left(\phi\left(\frac{-1+\sqrt{5}}{2}\right)\phi^{n-1} - \left(\hat{\phi}^2 - \hat{\phi}\right)\hat{\phi}^{n-1}\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\frac{-1+\sqrt{5}}{2}\phi^{n-1} - \left(\hat{\phi}^2 - \hat{\phi}\right)\hat{\phi}^{n-1}\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{-1+\sqrt{5}^2}{4}\phi^{n-1} - \left(\hat{\phi}^2 - \hat{\phi}\right)\hat{\phi}^{n-1}\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{4}{4}\phi^{n-1} - \left(\hat{\phi}^2 - \hat{\phi}\right)\hat{\phi}^{n-1}\right) \\
&= \frac{1}{\sqrt{5}}\left(\phi^{n-1} - \left(\hat{\phi}^2 - \hat{\phi}\right)\hat{\phi}^{n-1}\right) \\
&= \frac{1}{\sqrt{5}}\left(\phi^{n-1} - \hat{\phi}\left(\frac{1-\sqrt{5}}{2} - 1\right)\hat{\phi}^{n-1}\right) \\
&= \frac{1}{\sqrt{5}}\left(\phi^{n-1} - \hat{\phi}\left(\frac{-1-\sqrt{5}}{2}\right)\hat{\phi}^{n-1}\right) \\
&= \frac{1}{\sqrt{5}}\left(\phi^{n-1} - \frac{1-\sqrt{5}}{2}\frac{-1-\sqrt{5}}{2}\hat{\phi}^{n-1}\right) \\
&= \frac{1}{\sqrt{5}}\left(\phi^{n-1} - \frac{-1+\sqrt{5}^2}{4}\hat{\phi}^{n-1}\right) \\
&= \frac{1}{\sqrt{5}}\left(\phi^{n-1} - \frac{4}{4}\hat{\phi}^{n-1}\right) \\
&= \frac{1}{\sqrt{5}}\left(\phi^{n-1} - \hat{\phi}^{n-1}\right)
\end{aligned}
$$

as we needed to show. □

**Proposition.** *$F_n \neq \frac{\phi^n}{\sqrt{5}}$ rounded to the nearest integer for negative $n$.*

*Proof.* Consider $n = -1$. Then $F_{-1} = 1$ but since $\sqrt{5} > 2$,

$$\begin{aligned}\frac{\phi^{-1}}{\sqrt{5}} &= \frac{2}{1+\sqrt{5}}\frac{1}{\sqrt{5}} \\ &= \frac{2}{\sqrt{5}+5} \\ &< \frac{2}{2+5} \\ &= \frac{2}{7}\end{aligned}$$

rounded to the nearest integer is 0. □

**10.** [*15*] Is $\phi^n/\sqrt{5}$ greater than $F_n$ or less than $F_n$?

From Eq. (14),

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \hat{\phi}^n\right),$$

if and only if

$$\frac{\phi^n}{\sqrt{5}} - F_n = \frac{\hat{\phi}^n}{\sqrt{5}}.$$

That is, $\frac{\phi^n}{\sqrt{5}}$ is greater than $F_n$ when

$$\frac{\hat{\phi}^n}{\sqrt{5}} > 0 \qquad \Longleftrightarrow \qquad \hat{\phi}^n > 0$$

and less than $F_n$ when negative. Since

$$\begin{aligned}\sqrt{5} > 1 \quad &\Longleftrightarrow \quad 1-\sqrt{5} < 0 \\ &\Longleftrightarrow \quad \frac{1-\sqrt{5}}{2} < 0 \\ &\Longleftrightarrow \quad \hat{\phi} < 0,\end{aligned}$$

we have that $\hat{\phi}^n > 0$ when $n$ is even, negative when odd. That is, $\frac{\phi^n}{\sqrt{5}}$ is greater than $F_n$ when $n$ is even, less than $F_n$ when $n$ is odd.

**11.** [*M20*] Show that $\phi^n = F_n\phi + F_{n-1}$ and $\hat{\phi}^n = F_n\hat{\phi} + F_{n-1}$, for *all* integers $n$.

We show both identities.

**Proposition.** $\phi^n = F_n\phi + F_{n-1}$.

*Proof.* Let $n$ be an arbitrary integer. We must show that

$$\phi^n = F_n\phi + F_{n-1}.$$

We divide the proof into two cases: $n$ nonnegative, or $n$ nonpositive.

In the case that $n$ is nonnegative, if $n = 0$,

$$\phi^0 = 1 = 0 + 1 = F_0\phi + F_{-1};$$

and if $n = 1$,

$$\phi^1 = \phi + 0 = F_1\phi + F_0.$$

Then, assuming

$$\phi^n = F_n\phi + F_{n-1}$$

we must show that

$$\phi^{n+1} = F_{n+1}\phi + F_n.$$

But

$$\begin{aligned}\phi^{n+1} &= \phi\phi^n \\ &= \phi\left(F_n\phi + F_{n-1}\right) \\ &= F_n\phi^2 + F_{n-1}\phi \\ &= F_n\left(\phi + 1\right) + F_{n-1}\phi \\ &= F_n\phi + F_{n-1}\phi + F_n \\ &= \left(F_n + F_{n-1}\right)\phi + F_n \\ &= F_{n+1}\phi + F_n.\end{aligned}$$

In the case that $n$ is nonpositive, if $n = 0$,

$$\phi^0 = 1 = 0 + 1 = F_0\phi + F_{-1};$$

and if $n = -1$,

$$\phi^{-1} = \phi - 1 = F_{-1}\phi + F_{-2}.$$

Then, assuming

$$\phi^n = F_n\phi + F_{n-1}$$

we must show that

$$\phi^{n-1} = F_{n-1}\phi + F_{n-2}.$$

But

$$\begin{aligned}\phi^{n-1} &= \phi^{-1}\phi^n \\ &= \phi^{-1}\left(F_n\phi + F_{n-1}\right) \\ &= F_n + F_{n-1}\phi^{-1} \\ &= F_{n-1}\phi^{-1} + F_n \\ &= F_{n-1}\left(\phi - 1\right) + F_n \\ &= F_{n-1}\phi + F_n - F_{n-1} \\ &= F_{n-1}\phi + F_{n-2}.\end{aligned}$$

Therefore,

$$\phi^n = F_n\phi + F_{n-1}$$

for all integers $n$ as we needed to show. □

**Proposition.** $\hat{\phi}^n = F_n\hat{\phi} + F_{n-1}$.

*Proof.* Let $n$ be an arbitrary integer. We must show that

$$\hat{\phi}^n = F_n\hat{\phi} + F_{n-1}.$$

We divide the proof into two cases: $n$ nonnegative, or $n$ nonpositive.

In the case that $n$ is nonnegative, if $n = 0$,

$$\hat{\phi}^0 = 1 = 0 + 1 = F_0\hat{\phi} + F_{-1};$$

and if $n = 1$,

$$\hat{\phi}^1 = \hat{\phi} + 0 = F_1\hat{\phi} + F_0.$$

Then, assuming

$$\hat{\phi}^n = F_n\hat{\phi} + F_{n-1}$$

we must show that

$$\hat{\phi}^{n+1} = F_{n+1}\hat{\phi} + F_n.$$

But

$$\begin{aligned}
\hat{\phi}^{n+1} &= \hat{\phi}\hat{\phi}^n \\
&= \hat{\phi}\left(F_n\hat{\phi} + F_{n-1}\right) \\
&= F_n\hat{\phi}^2 + F_{n-1}\hat{\phi} \\
&= F_n\left(\hat{\phi} + 1\right) + F_{n-1}\hat{\phi} \\
&= F_n\hat{\phi} + F_{n-1}\hat{\phi} + F_n \\
&= \left(F_n + F_{n-1}\right)\hat{\phi} + F_n \\
&= F_{n+1}\hat{\phi} + F_n.
\end{aligned}$$

In the case that $n$ is nonpositive, if $n = 0$,

$$\hat{\phi}^0 = 1 = 0 + 1 = F_0\hat{\phi} + F_{-1};$$

and if $n = -1$,

$$\hat{\phi}^{-1} = \hat{\phi} - 1 = F_{-1}\hat{\phi} + F_{-2}.$$

Then, assuming

$$\hat{\phi}^n = F_n\hat{\phi} + F_{n-1}$$

we must show that

$$\hat{\phi}^{n-1} = F_{n-1}\hat{\phi} + F_{n-2}.$$

But

$$\begin{aligned}
\hat{\phi}^{n-1} &= \hat{\phi}^{-1}\hat{\phi}^n \\
&= \hat{\phi}^{-1}\left(F_n\hat{\phi} + F_{n-1}\right) \\
&= F_n + F_{n-1}\hat{\phi}^{-1} \\
&= F_{n-1}\hat{\phi}^{-1} + F_n \\
&= F_{n-1}\left(\hat{\phi} - 1\right) + F_n \\
&= F_{n-1}\hat{\phi} + F_n - F_{n-1} \\
&= F_{n-1}\hat{\phi} + F_{n-2}.
\end{aligned}$$

Therefore,

$$\hat{\phi}^n = F_n\hat{\phi} + F_{n-1}$$

for all integers $n$ as we needed to show. □

▶ **12.** [*M26*] The "second order" Fibonacci sequence is defined by the rule

$$\mathcal{F}_0 = 0, \qquad \mathcal{F}_1 = 1, \qquad \mathcal{F}_{n+2} = \mathcal{F}_{n+1} + \mathcal{F}_n + F_n.$$

Express $\mathcal{F}_n$ in terms of $F_n$ and $F_{n+1}$. [*Hint:* Use generating functions.]

Let

$$\mathcal{G}(z) = \sum \mathcal{F}_n z^n = \mathcal{F}_0 + \mathcal{F}_1 z + \mathcal{F}_2 z^2 + \cdots,$$
$$G(z) = \sum F_n z^n = F_0 + F_1 z + F_2 z^2 + \cdots,$$

and note that

$$F_n = \mathcal{F}_{n+2} - \mathcal{F}_{n+1} - \mathcal{F}_n.$$

Then

$$z\mathcal{G}(z) = \sum \mathcal{F}_n z^{n+1} = \mathcal{F}_0 z + \mathcal{F}_1 z^2 + \mathcal{F}_2 z^3 + \cdots,$$
$$z^2\mathcal{G}(z) = \sum \mathcal{F}_n z^{n+2} = \mathcal{F}_0 z^2 + \mathcal{F}_1 z^3 + \mathcal{F}_2 z^4 + \cdots,$$

and

$$\begin{aligned}
(1 - z - z^2)\mathcal{G}(z) &= \mathcal{F}_0 + (\mathcal{F}_1 - \mathcal{F}_0)z + \sum_{n\geq 2}(\mathcal{F}_n - \mathcal{F}_{n-1} - \mathcal{F}_{n-2})z^n \\
&= \mathcal{F}_0 + (\mathcal{F}_1 - \mathcal{F}_0)z + (\mathcal{F}_2 - \mathcal{F}_1 - \mathcal{F}_0)z^2 + \cdots \\
&= 0 + z + F_0 z^2 + \cdots \\
&= z + \sum F_n z^{n+2} \\
&= z + z^2 \sum F_n z^n \\
&= z + z^2 G(z).
\end{aligned}$$

From Eq. (11)

$$\frac{z}{G(z)}\mathcal{G}(z) = z + z^2 G(z)$$

if and only if by definition and from Eq. (17)

$$\begin{aligned}
\mathcal{G}(z) &= G(z) + zG^2(z) \\
&= \sum F_n z^n + z \sum \left(\frac{1}{2}(n-1)F_n + \frac{2}{5}nF_{n-1}\right) z^n \\
&= \sum F_{n+1} z^{n+1} + \sum \left(\frac{1}{2}(n-1)F_n + \frac{2}{5}nF_{n-1}\right) z^{n+1} \\
&= \sum \left(F_{n+1} + \frac{1}{2}(n-1)F_n + \frac{2}{5}nF_{n-1}\right) z^{n+1} \\
&= \sum \left(F_n + \frac{1}{2}(n-2)F_{n-1} + \frac{2}{5}(n-1)F_{n-2}\right) z^n.
\end{aligned}$$

But

$$\begin{aligned}
F_n &+ \frac{1}{2}(n-2)F_{n-1} + \frac{2}{5}(n-1)F_{n-2} \\
&= F_{n-1} + F_{n-2} + \frac{n-2}{5}F_{n-1} + \frac{2n-2}{5}F_{n-2} \\
&= \frac{n+3}{5}F_{n-1} + \frac{2n+3}{5}F_{n-2} \\
&= \frac{2n+3}{5}F_{n-1} + \frac{2n+3}{5}F_{n-2} - \frac{n}{5}F_{n-1} \\
&= \frac{2n+3}{5}F_n - \frac{n}{5}F_{n-1} \\
&= \frac{3n+3}{5}F_n - \frac{n}{5}F_n - \frac{n}{5}F_{n-1} \\
&= \frac{3n+3}{5}F_n - \frac{n}{5}F_{n+1}.
\end{aligned}$$

That is

$$\mathcal{F}_n = \frac{3n+3}{5}F_n - \frac{n}{5}F_{n+1}.$$

▶ **13.** [*M22*] Express the following sequences in terms of the Fibonacci numbers, when $r$, $s$, and $c$ are given constants.

a) $a_0 = r$, $a_1 = s$; $a_{n+2} = a_{n+1} + a_n$ for $n \geq 0$.

b) $b_0 = 0$, $b_1 = 1$; $b_{n+2} = b_{n+1} + b_n + c$, for $n \geq 0$.

We may express the sequences in terms of the Fibonacci numbers.

a) Allowing for negative $n$ so that $F_{-1} = 1$, we can express $a_n$ in terms of the Fibonacci numbers as

$$\begin{aligned}
a_0 &= r = sF_0 + rF_{-1}\\
a_1 &= s = sF_1 + rF_0\\
a_2 &= a_1 + a_0 = sF_1 + rF_0 + sF_0 + rF_{-1} = sF_2 + rF_1\\
&\cdots
\end{aligned}$$

and in general for $n \geq 0$ as

$$a_n = sF_n + rF_{n-1}.$$

We may prove this by induction. In the case that $n = 0$, $a_0 = sF_0 + rF_{-1}$; and in the case that $n = 1$, $a_1 = sF_1 + rF_0$. Then, assuming $a_n = sF_n + rF_{n-1}$, we must show that $a_{n+1} = sF_{n+1} + rF_n$. But

$$\begin{aligned}
a_{n+1} &= a_n + a_{n-1}\\
&= sF_n + rF_{n-1} + sF_{n-1} + rF_{n-2}\\
&= s\left(F_n + F_{n-1}\right) + r\left(F_{n-1} + F_{n-2}\right)\\
&= sF_{n+1} + rF_n
\end{aligned}$$

and hence the result.

b) We can express $b_n$ in terms of the Fibonacci numbers by first analyzing the derivative sequence $b'_n = b_n + c$ as

$$\begin{aligned}
b'_0 &= b_0 + c = 0 + c = c\\
b'_1 &= b_1 + c = 1 + c\\
&\cdots\\
b'_{n+2} &= b_{n+2} + c = b_{n+1} + b_n + c + c = b'_{n+1} + b'_n.
\end{aligned}$$

From (a) we have that

$$b'_n = (1+c)F_n + cF_{n-1}$$

if and only if

$$b_n = (1+c)F_n + cF_{n-1} - c$$

for $n \geq 0$.

**14.** [*M28*] Let $m$ be a fixed positive integer. Find $a_n$, given that

$$a_0 = 0, \qquad a_1 = 1; \qquad a_{n+2} = a_{n+1} + a_n + \binom{n}{m}, \quad \text{for } n \geq 0.$$

First, we note that for nonnegative integers $n \geq 0$

$$F_n = \sum_{0 \leq k \leq n-1} \binom{k}{n-k-1}, \tag{14.1}$$

which may be shown using induction, since

$$F_0 = 0 = \sum_{0 \leq k \leq -1} \binom{k}{-k-1}$$

and

$$F_1 = 1 = \sum_{0 \leq k \leq 0} \binom{k}{-k};$$

and assuming

$$F_n = \sum_{0 \leq k \leq n-1} \binom{k}{n-k-1}$$

implies

$$\begin{aligned}
F_{n+1} &= F_n + F_{n-1} \\
&= \sum_{0 \leq k \leq n-1} \binom{k}{n-k-1} + \sum_{0 \leq k \leq n-2} \binom{k}{n-k-2} \\
&= \binom{n-1}{0} + \sum_{0 \leq k \leq n-2} \binom{k}{n-k-1} + \sum_{0 \leq k \leq n-2} \binom{k}{n-k-2} \\
&= 1 + \sum_{0 \leq k \leq n-2} \binom{k}{n-k-1} + \sum_{0 \leq k \leq n-2} \binom{k}{n-k-2} \\
&= 1 + \sum_{0 \leq k \leq n-2} \binom{k+1}{n-k-1} \\
&= 1 + \sum_{1 \leq k \leq n-1} \binom{k}{n-k} \\
&= 1 + \sum_{0 \leq k \leq n-1} \binom{k}{n-k} - \binom{0}{n} \\
&= 1 + \sum_{0 \leq k \leq n-1} \binom{k}{n-k} \\
&= 1 + \sum_{0 \leq k \leq n} \binom{k}{n-k} - \binom{n}{0} \\
&= 1 + \sum_{0 \leq k \leq n} \binom{k}{n-k} - 1 \\
&= \sum_{0 \leq k \leq n} \binom{k}{n-k} \\
&= \sum_{0 \leq k \leq n-1+1} \binom{k}{n+1-k-1}.
\end{aligned}$$

Second, we note that for nonnegative integers $m, n \geq 0$

$$\sum_{0 \leq k \leq m} \left( \binom{n+k}{m-k} - \binom{n+k+1}{m-k-1} \right) = \binom{n}{m}, \tag{14.2}$$

which may be shown using induction, since

$$\binom{n}{0} - \binom{n+1}{-1} = 1 - 0 = 1 = \binom{n}{0}$$

and

$$\binom{n}{0} - \binom{n+1}{-1} + \binom{n}{1} - \binom{n+1}{0} = 1 + n - 1 = n = \binom{n}{1};$$

and assuming

$$\sum_{0 \le k \le m} \left( \binom{n+k}{m-k} - \binom{n+k+1}{m-k-1} \right) = \binom{n}{m},$$

including the induction basis $\binom{n-1}{n-m-2} = \binom{n-1}{n-1-(m+1)} = \binom{n-1}{m+1}$, implies

$$\begin{aligned}
&\sum_{0 \le k \le m+1} \left( \binom{n+k}{m+1-k} - \binom{n+k+1}{m+1-k-1} \right) \\
&\quad = \sum_{0 \le k \le m+1} \left( \binom{n+k}{m-k+1} - \binom{n+k+1}{m-k} \right) \\
&\quad = \left( \binom{n+(m+1)}{m-(m+1)+1} - \binom{n+(m+1)+1}{m-(m+1)} \right) + \sum_{0 \le k \le m} \left( \binom{n+k}{m-k+1} - \binom{n+k+1}{m-k} \right) \\
&\quad = \left( \binom{n+m+1}{0} - \binom{n+m+2}{-1} \right) + \sum_{0 \le k \le m} \left( \binom{n+k}{m-k+1} - \binom{n+k+1}{m-k} \right) \\
&\quad = (1-0) + \sum_{0 \le k \le m} \left( \binom{n+k}{m-k+1} - \binom{n+k+1}{m-k} \right) \\
&\quad = \left( \binom{n+m}{0} - \binom{n+m+1}{-1} \right) + \sum_{0 \le k \le m} \left( \binom{n+k}{m-k+1} - \binom{n+k+1}{m-k} \right) \\
&\quad = \left( \binom{n-1+(m+1)}{m+1-(m+1)} - \binom{n+(m+1)}{m-(m+1)} \right) \\
&\qquad + \sum_{0 \le k \le m} \left( \binom{n-1+k}{m+1-k} - \binom{n+k}{m-k} + \binom{n-1+k}{m-k} - \binom{n+k}{m-k-1} \right) \\
&\quad = \sum_{0 \le k \le m+1} \left( \binom{n-1+k}{m+1-k} - \binom{n+k}{m-k} \right) + \sum_{0 \le k \le m} \left( \binom{n-1+k}{m-k} - \binom{n+k}{m-k-1} \right) \\
&\quad = \sum_{0 \le k \le m+1} \left( \binom{n-1+k}{m+1-k} - \binom{n-1+k+1}{m+1-k-1} \right) + \sum_{0 \le k \le m} \left( \binom{n-1+k}{m-k} - \binom{n-1+k+1}{m-k-1} \right) \\
&\quad = \binom{n-1}{m+1} + \binom{n-1}{m}. \\
&\quad = \binom{n}{m+1}.
\end{aligned}$$

Finally, we claim that

$$a_n = F_{m+n+1} + F_n - \sum_{0 \le k \le m} \binom{n+k}{m-k}.$$

In the case that $n = 0$,

$$\begin{aligned}
a_0 &= 0 \\
&= F_{m+1} - \sum_{0\le k\le m+1-1} \binom{k}{m+1-k-1} && \text{from (14.1)} \\
&= F_{m+1} - \sum_{0\le k\le m} \binom{k}{m-k} \\
&= F_{m+1} + 0 - \sum_{0\le k\le m} \binom{0+k}{m-k} \\
&= F_{m+1} + F_0 - \sum_{0\le k\le m} \binom{0+k}{m-k};
\end{aligned}$$

and in the case that $n = 1$,

$$\begin{aligned}
a_1 &= 1 \\
&= F_1 \\
&= F_1 + 0 \\
&= F_1 + F_{m+2} - \sum_{0\le 1+k\le m+2-1} \binom{1+k}{m+2-(1+k)-1} && \text{from (14.1)} \\
&= F_{m+2} + F_1 - \sum_{-1\le k\le m} \binom{1+k}{m-k} \\
&= F_{m+2} + F_1 - \sum_{0\le k\le m} \binom{1+k}{m-k} - \binom{0}{m+1} \\
&= F_{m+2} + F_1 - \sum_{0\le k\le m} \binom{1+k}{m-k} - 0 \\
&= F_{m+2} + F_1 - \sum_{0\le k\le m} \binom{1+k}{m-k}.
\end{aligned}$$

Then, assuming

$$a_n = F_{m+n+1} + F_n - \sum_{0\le k\le m} \binom{n+k}{m-k},$$

we must show that

$$a_{n+1} = F_{m+n+2} + F_{n+1} - \sum_{0\le k\le m} \binom{n+k+1}{m-k}.$$

But

$$\begin{aligned}
a_{n+1} &= a_n + a_{n-1} + \binom{n-1}{m} \\
&= F_{m+n+1} + F_n - \sum_{0\le k\le m}\binom{n+k}{m-k} + F_{m+n} + F_{n-1} - \sum_{0\le k\le m}\binom{n+k-1}{m-k} + \binom{n-1}{m} \\
&= F_{m+n+2} + F_{n+1} - \sum_{0\le k\le m}\binom{n+k}{m-k} - \sum_{0\le k\le m}\binom{n+k-1}{m-k} + \binom{n-1}{m} \\
&= F_{m+n+2} + F_{n+1} - \sum_{0\le k\le m}\left(\binom{n+k}{m-k} + \binom{n+k-1}{m-k}\right) \\
&\quad + \sum_{0\le k\le m}\left(\binom{n-1+k}{m-k} - \binom{n-1+k+1}{m-k-1}\right) && \text{from (14.2)} \\
&= F_{m+n+2} + F_{n+1} + \sum_{0\le k\le m}\left(\binom{n+k-1}{m-k} - \binom{n+k}{m-k} - \binom{n+k-1}{m-k} - \binom{n+k}{m-k-1}\right) \\
&= F_{m+n+2} + F_{n+1} - \sum_{0\le k\le m}\left(\binom{n+k}{m-k} + \binom{n+k}{m-k-1}\right) \\
&= F_{m+n+2} + F_{n+1} - \sum_{0\le k\le m}\binom{n+k+1}{m-k}
\end{aligned}$$

and hence the result.

**15.** [*M22*] Let $f(n)$ and $g(n)$ be arbitrary functions, and for $n \ge 0$ let

$$\begin{aligned}
a_0 &= 0, & a_1 &= 1, & a_{n+2} &= a_{n+1} + a_n + f(n); \\
b_0 &= 0, & b_1 &= 1, & b_{n+2} &= b_{n+1} + b_n + g(n); \\
c_0 &= 0, & c_1 &= 1, & c_{n+2} &= c_{n+1} + c_n + xf(n) + yg(n).
\end{aligned}$$

Express $c_n$ in terms of $x$, $y$, $a_n$, $b_n$, and $F_n$.

We first prove two corollaries.

**Proposition.** $a_n = F_n + \sum_{1\le k\le n-1} F_k f(n-k-1)$.

*Proof.* Let $f(n)$ be an arbitrary function and $a_n$ defined as

$$a_{n+2} = a_{n+1} + a_n + f(n)$$

for $n \ge 0$ with $a_1 = 1$ and $a_0 = 0$. We will show that

$$a_n = F_n + \sum_{1\le k\le n-1} F_k f(n-k-1). \tag{15.1}$$

In the case that $n = 0$,

$$a_0 = 0 = F_0 + \sum_{1\le k\le -1} F_k f(n-k-1);$$

and in the case that $n = 1$,

$$a_1 = 1 = F_1 + \sum_{1\le k\le 0} F_k f(n-k-1).$$

Then, assuming

$$a_n = F_n + \sum_{1 \le k \le n-1} F_k f(n-k-1)$$

we must show that

$$a_{n+1} = F_{n+1} + \sum_{1 \le k \le n} F_k f(n-k).$$

But

$$\begin{aligned}
&a_{n+1}\\
&\quad = a_n + a_{n-1} + f(n-1)\\
&\quad = F_n + \sum_{1 \le k \le n-1} F_k f(n-k-1) + F_{n-1} + \sum_{1 \le k \le n-2} F_k f(n-k-2) + f(n-1)\\
&\quad = F_n + F_{n-1} + f(n-1) + \sum_{1 \le k \le n-1} F_k f(n-k-1) + \sum_{1 \le k \le n-2} F_k f(n-k-2)\\
&\quad = F_{n+1} + f(n-1) + \sum_{1 \le k \le n-1} F_k f(n-k-1) + \sum_{1 \le k \le n-2} F_k f(n-k-2)\\
&\quad = F_{n+1} + f(n-1) + \sum_{2 \le k \le n} F_{k-1} f(n-k) + \sum_{3 \le k \le n} F_{k-2} f(n-k)\\
&\quad = F_{n+1} + f(n-1) + \sum_{2 \le k \le n} F_{k-1} f(n-k) + \sum_{2 \le k \le n} F_{k-2} f(n-k)\\
&\quad = F_{n+1} + f(n-1) + \sum_{2 \le k \le n} \left(F_{k-1} + F_{k-2}\right) f(n-k)\\
&\quad = F_{n+1} + f(n-1) + \sum_{2 \le k \le n} F_k f(n-k)\\
&\quad = F_{n+1} + \sum_{1 \le k \le n} F_k f(n-k)
\end{aligned}$$

and hence the result. □

**Proposition.** $c_n = F_n + x \sum_{1 \le k \le n-1} F_k f(n-k-1) + y \sum_{1 \le k \le n-1} F_k g(n-k-1)$.

*Proof.* Let $f(n)$ and $g(n)$ be arbitrary functions; and $a_n$, $b_n$, $c_n$ defined as

$$\begin{aligned}
a_0 &= 0, & a_1 &= 1, & a_{n+2} &= a_{n+1} + a_n + f(n);\\
b_0 &= 0, & b_1 &= 1, & b_{n+2} &= b_{n+1} + b_n + g(n);\\
c_0 &= 0, & c_1 &= 1, & c_{n+2} &= c_{n+1} + c_n + xf(n) + yg(n).
\end{aligned}$$

We will show that

$$c_n = F_n + x \sum_{1 \le k \le n-1} F_k f(n-k-1) + y \sum_{1 \le k \le n-1} F_k g(n-k-1). \tag{15.2}$$

In the case that $n = 0$,

$$c_0 = 0 = F_0 + x \sum_{1 \le k \le -1} F_k f(n-k-1) + y \sum_{1 \le k \le -1} F_k g(n-k-1);$$

and in the case that $n = 1$,

$$c_1 = 1 = F_1 + x \sum_{1 \le k \le 0} F_k f(n-k-1) + y \sum_{1 \le k \le 0} F_k f(n-k-1).$$

Then, assuming

$$c_n = F_n + x \sum_{1 \le k \le n-1} F_k f(n-k-1) + y \sum_{1 \le k \le n-1} F_k g(n-k-1)$$

we must show that

$$c_{n+1} = F_{n+1} + x \sum_{1 \le k \le n} F_k f(n-k) + y \sum_{1 \le k \le n} F_k g(n-k).$$

But

$$\begin{aligned}
&c_{n+1} \\
&\quad = c_n + c_{n-1} + xf(n-1) + yg(n-1) \\
&\quad = F_n + x \sum_{1 \le k \le n-1} F_k f(n-k-1) + y \sum_{1 \le k \le n-1} F_k g(n-k-1) \\
&\qquad + F_{n-1} + x \sum_{1 \le k \le n-2} F_k f(n-k-2) + y \sum_{1 \le k \le n-2} F_k g(n-k-2) \\
&\qquad + xf(n-1) + yg(n-1) \\
&\quad = F_{n+1} + xf(n-1) + yg(n-1) \\
&\qquad + x \sum_{1 \le k \le n-1} F_k f(n-k-1) + y \sum_{1 \le k \le n-1} F_k g(n-k-1) \\
&\qquad + x \sum_{1 \le k \le n-2} F_k f(n-k-2) + y \sum_{1 \le k \le n-2} F_k g(n-k-2) \\
&\quad = F_{n+1} + xf(n-1) + yg(n-1) \\
&\qquad + x \sum_{2 \le k \le n} F_{k-1} f(n-k) + y \sum_{2 \le k \le n} F_{k-1} g(n-k) \\
&\qquad + x \sum_{3 \le k \le n} F_{k-2} f(n-k) + y \sum_{3 \le k \le n} F_{k-2} g(n-k) \\
&\quad = F_{n+1} + xf(n-1) + yg(n-1) \\
&\qquad + x \sum_{2 \le k \le n} F_{k-1} f(n-k) + y \sum_{2 \le k \le n} F_{k-1} g(n-k) \\
&\qquad + x \sum_{2 \le k \le n} F_{k-2} f(n-k) + y \sum_{2 \le k \le n} F_{k-2} g(n-k) \\
&\quad = F_{n+1} + xf(n-1) + yg(n-1) \\
&\qquad + x \sum_{2 \le k \le n} \left(F_{k-1} + F_{k-2}\right) f(n-k) \\
&\qquad + y \sum_{2 \le k \le n} \left(F_{k-1} + F_{k-2}\right) g(n-k) \\
&\quad = F_{n+1} + xf(n-1) + yg(n-1) + x \sum_{2 \le k \le n} F_k f(n-k) + y \sum_{2 \le k \le n} F_k g(n-k) \\
&\quad = F_{n+1} + x \sum_{1 \le k \le n} F_k f(n-k) + y \sum_{1 \le k \le n} F_k g(n-k)
\end{aligned}$$

and hence the result. □

We then solve for $c_n$ as

$$\begin{aligned} c_n &= F_n + x \sum_{1\le k\le n-1} F_k f(n-k-1) + y \sum_{1\le k\le n-1} F_k g(n-k-1) && \text{from (15.2)} \\ &= F_n + x\,(a_n - F_n) + y\,(b_n - F_n) && \text{from (15.1)} \\ &= xa_n + yb_n + F_n - xF_n - yF_n \\ &= xa_n + yb_n + (1-x-y)F_n. \end{aligned}$$

▶ **16.** [*M20*] Fibonacci numbers appear implicitly in Pascal's triangle if it is viewed from the right angle. Show that the following sum of binomial coefficients is a Fibonacci number:

$$\sum_{k=0}^{n} \binom{n-k}{k}.$$

We may prove that the sum is a Fibonacci number.

**Proposition.** $\sum_{0\le k\le n} \binom{n-k}{k} = F_{n+1}$.

*Proof.* Let $n$ be an arbitrary nonnegative integer such that $n \ge 0$. We must show that

$$\sum_{0\le k\le n} \binom{n-k}{k} = F_{n+1}.$$

In the case that $n = 0$,

$$\sum_{0\le k\le 0} \binom{-k}{k} = \binom{0}{0} = 1 = F_1;$$

and in the case that $n = 1$,

$$\sum_{0\le k\le 1} \binom{1-k}{k} = \binom{1}{0} + \binom{0}{1} = 1 + 0 = F_2.$$

Then, assuming

$$\sum_{0\le k\le n} \binom{n-k}{k} = F_{n+1},$$

we must show that

$$\sum_{0\le k\le n+1} \binom{n+1-k}{k} = F_{n+2}.$$

But

$$\begin{aligned}
&\sum_{0\le k\le n+1}\binom{n+1-k}{k}\\
&= \binom{0}{n+1} + \sum_{0\le k\le n}\binom{n+1-k}{k}\\
&= \sum_{0\le k\le n}\binom{n+1-k}{k}\\
&= \sum_{0\le k\le n}\left(\binom{n-k}{k}+\binom{n-k}{k-1}\right)\\
&= \sum_{0\le k\le n}\binom{n-k}{k} + \sum_{0\le k\le n}\binom{(n-1)-(k-1)}{k-1}\\
&= F_{n+1} + \sum_{0\le k\le n}\binom{(n-1)-(k-1)}{k-1}\\
&= F_{n+1} + \sum_{-1\le k\le n-1}\binom{n-1-k}{k}\\
&= F_{n+1} + \binom{n}{-1} + \sum_{0\le k\le n-1}\binom{n-1-k}{k}\\
&= F_{n+1} + \sum_{0\le k\le n-1}\binom{n-1-k}{k}\\
&= F_{n+1} + F_n\\
&= F_{n+2}
\end{aligned}$$

as we needed to show. □

**17.** [*M24*] Using the conventions of exercise 8, prove the following generalization of Eq. (4): $F_{n+k}F_{m-k} - F_nF_m = (-1)^n F_{m-n-k}F_k$.

We may prove the generalization, but first, a corollary.

**Proposition.** $(x^{n+k} - y^{n+k})(x^{m-k} - y^{m-k}) - (x^n - y^n)(x^m - y^m) = (xy)^n(x^{m-n-k} - y^{m-n-k})(x^k - y^k)$.

*Proof.* Let $x, y$ be arbitrary reals and $m, n, k$ arbitrary integers. We must show that

$$\begin{aligned}
(x^{n+k} - y^{n+k})(x^{m-k} - y^{m-k}) - (x^n - y^n)(x^m - y^m)&\\
= (xy)^n(x^{m-n-k} - y^{m-n-k})(x^k - y^k).&
\end{aligned} \tag{17.1}$$

But

$$\begin{aligned}
&(x^{n+k} - y^{n+k})(x^{m-k} - y^{m-k}) - (x^n - y^n)(x^m - y^m)\\
&\quad= (xy)^n((x^k y^{-n} - x^{-n}y^k)(x^{m-n-k}y^{-n} - x^{-n}y^{m-n-k})\\
&\qquad - (y^{-n} - x^{-n})(x^{m-n}y^{-n} - x^{-n}y^{m-n}))\\
&\quad= -x^{m-k}y^{n+k} - x^{n+k}y^{m-k} + x^m y^n + x^n y^m\\
&\quad= (xy)^n(-x^{m-n-k}y^k - x^k y^{m-n-k} + x^{m-n} + y^{m-n})\\
&\quad= (xy)^n(x^{m-n-k} - y^{m-n-k})(x^k - y^k)
\end{aligned}$$

and hence the result. □

Finally, the proof.

**Proposition.** $F_{n+k}F_{m-k} - F_nF_m = (-1)^n F_{m-n-k}F_k$.

*Proof.* Let $m$, $n$, $k$ be arbitrary integers, and allow for negative indexed Fibonacci numbers so that $F_{-n} = (-1)^{n+1}F_n$. We must show that

$$F_{n+k}F_{m-k} - F_nF_m = (-1)^n F_{m-n-k}F_k.$$

But since $\phi^n = (1-\hat{\phi})^n = -\hat{\phi}^{-n}$,

$$\begin{aligned}
&F_{n+k}F_{m-k} - F_nF_m \\
&\quad = \frac{1}{\sqrt{5}}\left(\phi^{n+k} - \hat{\phi}^{n+k}\right)\frac{1}{\sqrt{5}}\left(\phi^{m-k} - \hat{\phi}^{m-k}\right) - \frac{1}{\sqrt{5}}\left(\phi^{n} - \hat{\phi}^{n}\right)\frac{1}{\sqrt{5}}\left(\phi^{m} - \hat{\phi}^{m}\right) \\
&\quad = \frac{1}{\sqrt{5}^2}\left(\left(\phi^{n+k} - \hat{\phi}^{n+k}\right)\left(\phi^{m-k} - \hat{\phi}^{m-k}\right) - \left(\phi^{n} - \hat{\phi}^{n}\right)\left(\phi^{m} - \hat{\phi}^{m}\right)\right) \\
&\quad = \frac{1}{\sqrt{5}^2}\left(\phi\hat{\phi}\right)^n\left(\phi^{m-n-k} - \hat{\phi}^{m-n-k}\right)\left(\phi^{k} - \hat{\phi}^{k}\right) && \text{from (17.1)} \\
&\quad = \left(\phi\hat{\phi}\right)^n\frac{1}{\sqrt{5}}\left(\phi^{m-n-k} - \hat{\phi}^{m-n-k}\right)\frac{1}{\sqrt{5}}\left(\phi^{k} - \hat{\phi}^{k}\right) \\
&\quad = \left(\phi\hat{\phi}\right)^n F_{m-n-k}F_k \\
&\quad = \left(\frac{1}{2}(1+\sqrt{5})\frac{1}{2}(1-\sqrt{5})\right)^n F_{m-n-k}F_k \\
&\quad = \left(\frac{1}{4}(1+\sqrt{5}-\sqrt{5}-\sqrt{5}^2)\right)^n F_{m-n-k}F_k \\
&\quad = \left(\frac{-4}{4}\right)^n F_{m-n-k}F_k \\
&\quad = (-1)^n F_{m-n-k}F_k
\end{aligned}$$

as we needed to show. □

**18.** [*20*] Is $F_n^2 + F_{n+1}^2$ always a Fibonacci number?

Yes, $F_{2n+1}$, as shown below.

**Proposition.** $F_n^2 + F_{n+1}^2 = F_{2n+1}$.

*Proof.* Let $n$ be an arbitrary nonnegative integer. We must show that

$$F_n^2 + F_{n+1}^2 = F_{2n+1}.$$

In the case that $n = 0$,

$$F_0^2 + F_1^2 = 0 + 1 = 1 = F_1;$$

and in the case that $n = 1$,

$$F_1^2 + F_2^2 = 1 + 1 = 2 = F_3.$$

Then, assuming

$$F_n^2 + F_{n+1}^2 = F_{2n+1}.$$

we must show that

$$F_{n+1}^2 + F_{n+2}^2 = F_{2n+3}.$$

But

$$\begin{aligned}
F_{n+1}^2 + F_{n+2}^2 &= (F_n + F_{n-1})^2 + (F_{n+1} + F_n)^2 \\
&= F_n^2 + 2F_nF_{n-1} + F_{n-1}^2 + F_{n+1}^2 + 2F_{n+1}F_n + F_n^2 \\
&= F_{n-1}^2 + F_n^2 + F_n^2 + F_{n+1}^2 + 2F_nF_{n-1} + 2F_{n+1}F_n \\
&= F_{2n-1} + F_{2n+1} + 2F_nF_{n-1} + 2F_{n+1}F_n \\
&= F_{2n-1} + F_{2n+1} + 2F_nF_{n-1} + 2(F_{n+n} - F_{n-1}F_n) \qquad \text{from Eq. (6)} \\
&= F_{2n-1} + F_{2n+1} + 2F_nF_{n-1} + 2F_{2n} - 2F_{n-1}F_n \\
&= F_{2n-1} + F_{2n+1} + 2F_{2n} \\
&= F_{2n+1} + F_{2n} + F_{2n+1} \\
&= F_{2n+2} + F_{2n+1} \\
&= F_{2n+3}
\end{aligned}$$

as we needed to show. □

▶ **19.** [*M27*] What is $\cos 36°$?

We have that

$$\cos 36° = \frac{1+\sqrt{5}}{4},$$

derived as follows. From the double angle formulas we have that

$$\begin{aligned}
\cos 72° &= \cos(2 \cdot 36°) \\
&= 2\cos^2 36° - 1
\end{aligned}$$

and

$$\begin{aligned}
\cos 36° &= \cos(2 \cdot 18°) \\
&= 1 - 2\sin^2 18° \\
&= 1 - 2\sin^2(90° - 72°) \\
&= 1 - 2\cos^2 72°.
\end{aligned}$$

That is, that

$$\begin{aligned}
\cos 72° + \cos 36° &= 2\cos^2 36° - 1 + 1 - 2\cos^2 72° \\
&= 2\left(\cos^2 36° - \cos^2 72°\right)
\end{aligned}$$

if and only if

$$\begin{aligned}
1 &= \frac{2\left(\cos^2 36° - \cos^2 72°\right)}{\cos 72° + \cos 36°} \\
&= \frac{2\left(\cos 36° - \cos 72°\right)\left(\cos^2 72° + \cos 36°\right)}{\cos 72° + \cos 36°} \\
&= 2\left(\cos 36° - \cos 72°\right) \\
&= 2\cos 36° - 2\cos 72° \\
&= 2\cos 36° - 2\left(2\cos^2 36° - 1\right) \\
&= 2\cos 36° - 4\cos^2 36° + 2;
\end{aligned}$$

or equivalently that

$$2\cos 36^\circ + 1 = (2\cos 36^\circ)^2 .$$

And so, in terms of the golden ratio, since $2\cos 36^\circ = \phi$,

$$\begin{aligned}\cos 36^\circ &= \frac{1}{2}\phi \\ &= \frac{1}{2}\frac{1+\sqrt{5}}{2} \\ &= \frac{1+\sqrt{5}}{4}\end{aligned}$$

and hence the result.

**20.** [*M16*] Express $\sum_{k=0}^{n} F_k$ in terms of Fibonacci numbers.

We have that

$$\sum_{k=0}^{n} F_k = F_{n+2} - 1,$$

as shown here. In the case that $n = 0$,

$$\sum_{k=0}^{0} F_k = F_0 = 0 = 1 - 1 = F_2 - 1;$$

Then, assuming

$$\sum_{k=0}^{n} F_k = F_{n+2} - 1$$

we must show that

$$\sum_{k=0}^{n+1} F_k = F_{n+3} - 1.$$

But

$$\begin{aligned}\sum_{k=0}^{n+1} F_k &= \sum_{k=0}^{n} F_k + F_{n+1} \\ &= F_{n+2} - 1 + F_{n+1} \\ &= F_{n+3} - 1\end{aligned}$$

and hence the result.

**21.** [*M25*] What is $\sum_{k=0}^{n} F_k x^k$?

We have that

$$\sum_{k=0}^{n} F_k x^k = \begin{cases} \frac{x^{n+1}F_{n+1} + x^{n+2}F_n - x}{x^2+x-1} & \text{if } x^2 + x \neq 1 \\ \frac{n+1-x^n F_{n+1}}{2x+1} & \text{otherwise} \end{cases}$$

as shown here. First we consider the case that $x^2 + x \neq 1$. For $n = 0$,

$$\sum_{k=0}^{0} F_k x^k = F_0 x^0 = 0 = \frac{0}{x^2+x-1} = \frac{x+0-x}{x^2+x-1} = \frac{x^1 F_1 + x^2 F_0 - x}{x^2+x-1}.$$

Then, assuming

$$\sum_{k=0}^{n} F_k x^k = \frac{x^{n+1}F_{n+1} + x^{n+2}F_n - x}{x^2+x-1},$$

we must show that

$$\sum_{k=0}^{n+1} F_k x^k = \frac{x^{n+2}F_{n+2} + x^{n+3}F_{n+1} - x}{x^2+x-1}.$$

But

$$\begin{aligned}
&\sum_{k=0}^{n+1} F_k x^k \\
&= F_{n+1}x^{n+1} + \sum_{k=0}^{n} F_k x^k \\
&= F_{n+1}x^{n+1} + \frac{x^{n+1}F_{n+1} + x^{n+2}F_n - x}{x^2+x-1} \\
&= \frac{(x^2+x-1)F_{n+1}x^{n+1}}{x^2+x-1} + \frac{x^{n+1}F_{n+1} + x^{n+2}F_n - x}{x^2+x-1} \\
&= \frac{x^{n+3}F_{n+1} + x^{n+2}F_{n+1} - x^{n+1}F_{n+1}}{x^2+x-1} + \frac{x^{n+1}F_{n+1} + x^{n+2}F_n - x}{x^2+x-1} \\
&= \frac{x^{n+3}F_{n+1} + x^{n+2}F_{n+1} - x^{n+1}F_{n+1} + x^{n+1}F_{n+1} + x^{n+2}F_n - x}{x^2+x-1} \\
&= \frac{x^{n+2}F_{n+1} + x^{n+2}F_n + x^{n+3}F_{n+1} - x}{x^2+x-1} \\
&= \frac{x^{n+2}\left(F_{n+1} + F_n\right) + x^{n+3}F_{n+1} - x}{x^2+x-1} \\
&= \frac{x^{n+2}F_{n+2} + x^{n+3}F_{n+1} - x}{x^2+x-1}.
\end{aligned}$$

Last we consider the case that $x^2 + x = 1$. Note that in general since $x^{n+2} = x^n - x^{n+1}$,

$$1 = x^n F_{n+1} + x^{n+1} F_n, \tag{21.1}$$

since $1 = 1 + 0 = x^0 F_1 + x^1 F_0$ and $1 = x^n F_{n+1} + x^{n+1} F_n \implies 1 = x^{n+1} F_{n+2} + x^{n+2} F_{n+1}$ as

$$\begin{aligned}
x^{n+1}F_{n+2} + x^{n+2}F_{n+1} &= x^{n+1}F_{n+2} + \left(x^n - x^{n+1}\right)F_{n+1} \\
&= x^{n+1}F_{n+2} + x^nF_{n+1} - x^{n+1}F_{n+1} \\
&= x^{n+1}F_{n+2} - x^{n+1}F_{n+1} + x^nF_{n+1} \\
&= x^{n+1}\left(F_{n+2} - F_{n+1}\right) + x^nF_{n+1} \\
&= x^{n+1}F_n + x^nF_{n+1} \\
&= x^nF_{n+1} + x^{n+1}F_n \\
&= 1.
\end{aligned}$$

Again, considering the case that $x^2 + x = 1$, for $n = 0$,

$$\sum_{k=0}^{0} F_k x^k = F_0 x^0 = 0 = \frac{1-1}{2x+1} = \frac{1-F_1}{2x+1} = \frac{0+1-x^0F_1}{2x+1}.$$

Then, assuming

$$\sum_{k=0}^{n} F_k x^k = \frac{n+1-x^nF_{n+1}}{2x+1},$$

we must show that

$$\sum_{k=0}^{n+1} F_k x^k = \frac{n+2-x^{n+1}F_{n+2}}{2x+1}.$$

But in this case,

$$
\begin{aligned}
&\sum_{k=0}^{n+1} F_k x^k \\
&\quad = F_{n+1}x^{n+1} + \sum_{k=0}^{n} F_k x^k \\
&\quad = F_{n+1}x^{n+1} + \frac{n+1-x^n F_{n+1}}{2x+1} \\
&\quad = \frac{(2x+1)F_{n+1}x^{n+1}}{2x+1} + \frac{n+1-x^n F_{n+1}}{2x+1} \\
&\quad = \frac{2xF_{n+1}x^{n+1} + F_{n+1}x^{n+1}}{2x+1} + \frac{n+1-x^n F_{n+1}}{2x+1} \\
&\quad = \frac{2F_{n+1}x^{n+2} + F_{n+1}x^{n+1} + n + 1 - x^n F_{n+1}}{2x+1} \\
&\quad = \frac{n+2+2F_{n+1}x^{n+2} + F_{n+1}x^{n+1} - 1 - x^n F_{n+1}}{2x+1} \\
&\quad = \frac{n+2+2F_{n+1}(x^n - x^{n+1}) + F_{n+1}x^{n+1} - 1 - x^n F_{n+1}}{2x+1} \\
&\quad = \frac{n+2+2F_{n+1}x^n - 2F_{n+1}x^{n+1} + F_{n+1}x^{n+1} - 1 - x^n F_{n+1}}{2x+1} \\
&\quad = \frac{n+2-\bigl(x^{n+1}F_{n+1} - F_{n+1}x^n + 1\bigr)}{2x+1} \\
&\quad = \frac{n+2-\bigl(x^{n+1}F_{n+1} - F_{n+1}x^n + x^n F_{n+1} + x^{n+1}F_n\bigr)}{2x+1} && \text{from (21.1)} \\
&\quad = \frac{n+2-\bigl(x^{n+1}F_{n+1} + x^{n+1}F_n\bigr)}{2x+1} \\
&\quad = \frac{n+2-x^{n+1}\left(F_{n+1}+F_n\right)}{2x+1} \\
&\quad = \frac{n+2-x^{n+1}F_{n+2}}{2x+1}.
\end{aligned}
$$

Hence the result in either case.

▶ **22.** [*M20*] Show that $\sum_k \binom{n}{k} F_{m+k}$ is a Fibonacci number.

We have, by the binomial theorem, and since $1+\phi=\phi^2$ and $1+\hat{\phi}=\hat{\phi}^2$,

$$\begin{aligned}
\sum_k \binom{n}{k} F_{m+k} &= \sum_k \binom{n}{k} \frac{1}{\sqrt{5}} \left(\phi^{m+k} - \hat{\phi}^{m+k}\right) \\
&= \frac{1}{\sqrt{5}} \left(\phi^m \sum_k \binom{n}{k} \phi^k - \hat{\phi}^m \sum_k \binom{n}{k} \hat{\phi}^k\right) \\
&= \frac{1}{\sqrt{5}} \left(\phi^m (1+\phi)^n - \hat{\phi}^m \left(1+\hat{\phi}\right)^n\right) \\
&= \frac{1}{\sqrt{5}} \left(\phi^m \left(\phi^2\right)^n - \hat{\phi}^m \left(\hat{\phi}^2\right)^n\right) \\
&= \frac{1}{\sqrt{5}} \left(\phi^m \phi^{2n} - \hat{\phi}^m \hat{\phi}^{2n}\right) \\
&= \frac{1}{\sqrt{5}} \left(\phi^{m+2n} - \hat{\phi}^{m+2n}\right) \\
&= F_{m+2n}.
\end{aligned}$$

**23.** [*M23*] Generalizing the preceding exercise, show that $\sum_k \binom{n}{k} F_t^k F_{t-1}^{n-k} F_{m+k}$ is always a Fibonacci number.

First, a corollary.

**Proposition.** *$F_n\phi + F_{n-1} = \phi^n$ and $F_n\hat{\phi} + F_{n-1} = \hat{\phi}^n$.*

*Proof.* Let $n$ be an arbitrary, nonnegative integer. We must show that both

$$F_n\phi + F_{n-1} = \phi^n \tag{23.1}$$

and

$$F_n\hat{\phi} + F_{n-1} = \hat{\phi}^n. \tag{23.2}$$

In the case that $n=1$,

$$F_1\phi + F_{1-1} = F_1\phi + F_0 = \phi + 0 = \phi = \phi^1;$$

and in the case that $n=2$,

$$F_2\phi + F_{2-1} = F_2\phi + F_1 = \phi + 1 = \phi^2.$$

Then, assuming

$$F_n\phi + F_{n-1} = \phi^n$$

we must show that

$$F_{n+1}\phi + F_n = \phi^{n+1}.$$

But

$$\begin{aligned}
F_{n+1}\phi + F_n &= (F_n + F_{n-1})\,\phi + F_{n-1} + F_{n-2} \\
&= F_n\phi + F_{n-1} + F_{n-1}\phi + F_{n-2} \\
&= \phi^n + \phi^{n-1} \\
&= \phi^{n+1}
\end{aligned}$$

and hence the result for $\phi$. The result for $\hat\phi$ follows similarly as $F_1\hat\phi + F_{1-1} = F_1\hat\phi + F_0 = \hat\phi + 0 = \hat\phi = \hat\phi^1$, $F_2\hat\phi + F_{2-1} = F_2\hat\phi + F_1 = \hat\phi + 1 = \hat\phi^2$, and $F_n\hat\phi + F_{n-1} = \hat\phi^n \implies F_{n+1}\hat\phi + F_n = \hat\phi^{n+1}$ since

$$\begin{aligned} F_{n+1}\hat\phi + F_n &= (F_n + F_{n-1})\,\hat\phi + F_{n-1} + F_{n-2} \\ &= F_n\hat\phi + F_{n-1} + F_{n-1}\hat\phi + F_{n-2} \\ &= \hat\phi^n + \hat\phi^{n-1} \\ &= \hat\phi^{n+1}, \end{aligned}$$

as we needed to show. □

Then we have, by the binomial theorem, and by both (23.1) and (23.2),

$$\begin{aligned} \sum_k \binom{n}{k} F_t^k F_{t-1}^{n-k} F_{m+k} &= \sum_k \binom{n}{k} F_t^k F_{t-1}^{n-k} \frac{1}{\sqrt5}\left(\phi^{m+k} - \hat\phi^{m+k}\right) \\ &= \frac{1}{\sqrt5}\sum_k \binom{n}{k} F_t^k F_{t-1}^{n-k}\left(\phi^m\phi^k - \hat\phi^m\hat\phi^k\right) \\ &= \frac{1}{\sqrt5}\left(\phi^m \sum_k \binom{n}{k}\phi^k F_t^k F_{t-1}^{n-k} - \hat\phi^m \sum_k \binom{n}{k}\hat\phi^k F_t^k F_{t-1}^{n-k}\right) \\ &= \frac{1}{\sqrt5}\left(\phi^m \sum_k \binom{n}{k}(\phi F_t)^k F_{t-1}^{n-k} - \hat\phi^m \sum_k \binom{n}{k}\left(\hat\phi F_t\right)^k F_{t-1}^{n-k}\right) \\ &= \frac{1}{\sqrt5}\left(\phi^m (F_t\phi + F_{t-1})^n - \hat\phi^m\left(F_t\hat\phi + F_{t-1}\right)^n\right) \\ &= \frac{1}{\sqrt5}\left(\phi^m \left(\phi^t\right)^n - \hat\phi^m\left(\hat\phi^t\right)^n\right) \\ &= \frac{1}{\sqrt5}\left(\phi^m\phi^{tn} - \hat\phi^m\hat\phi^{tn}\right) \\ &= \frac{1}{\sqrt5}\left(\phi^{m+tn} - \hat\phi^{m+tn}\right) \\ &= F_{m+tn}. \end{aligned}$$

**24.** [*HM20*] Evaluate the $n \times n$ determinant

$$\begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix}.$$

Given

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 1 & \text{if } i = j+1 \\ -1 & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

we want to find $\det[a_{ij}]_n$. In the case that $n = 1$,

$$\det[a_{ij}]_1 = \begin{bmatrix} 1 \end{bmatrix} = 1 = F_2 = F_{1+1};$$

and in the case that $n = 2$,

$$\det[a_{ij}]_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = 1 \cdot 1 - (-1) \cdot 1 = 1 + 1 = 2 = F_3 = F_{2+1}.$$

Then, assuming

$$\det[a_{ij}]_n = F_{n+1}$$

we need to show that

$$\det[a_{ij}]_{n+1} = F_{n+2}$$

But

$$\begin{aligned}
\det[a_{ij}]_{n+1} &= \sum_{1 \le j \le n+1} a_{1j} \cdot \text{cofactor}(a_{1j}) \\
&= a_{11} \cdot \text{cofactor}(a_{11}) + a_{12} \cdot \text{cofactor}(a_{12}) + \sum_{3 \le j \le n+1} a_{1j} \cdot \text{cofactor}(a_{1j}) \\
&= \text{cofactor}(a_{11}) - \text{cofactor}(a_{12}) + 0 \\
&= (-1)^{1+1} \det \text{minor}([a]_{n+1}, 1, 1) - (-1)^{1+2} \det \text{minor}([a]_{n+1}, 1, 2) \\
&= \det \text{minor}([a]_{n+1}, 1, 1) + \det \text{minor}([a]_{n+1}, 1, 2).
\end{aligned}$$

Note that $\text{minor}([a]_{n+1}, 1, 1)$ preserves symmetry about the diagonal so that

$$\text{minor}([a]_{n+1}, 1, 1) = [a]_n; \qquad (24.1)$$

and for

$$a'_{ij} = \begin{cases} a_{ij} & \text{if } i \ne 2 \lor j \ne 1 \\ 0 & \text{otherwise} \end{cases},$$

that $\det \text{minor}([a]_{n+1}, 1, 2)$ can be expanded further as

$$\begin{aligned}
\det \text{minor}([a]_{n+1}, 1, 2) &= \det[a']_n \\
&= \sum_{1 \le i \le n} a'_{i1} \cdot \text{cofactor}(a'_{i1}) \\
&= a'_{11} \cdot \text{cofactor}(a'_{11}) + \sum_{2 \le i \le n} a'_{i1} \cdot \text{cofactor}(a'_{i1}) \\
&= a_{11} \cdot \text{cofactor}(a'_{11}) + 0 \\
&= \text{cofactor}(a'_{11}) \\
&= (-1)^{1+1} \det \text{minor}([a']_n, 1, 1) \\
&= \det[a]_{n-1};
\end{aligned}$$

that is, that

$$\det \text{minor}([a]_{n+1}, 1, 2) = \det[a]_{n-1}. \qquad (24.2)$$

And so,

$$\begin{aligned}
\det[a_{ij}]_{n+1} &= \det \text{minor}([a]_{n+1}, 1, 1) + \det \text{minor}([a]_{n+1}, 1, 2) \\
&= \det[a]_n + \det \text{minor}([a]_{n+1}, 1, 2) && \text{by (24.1)} \\
&= \det[a]_n + \det[a]_{n-1} && \text{by (24.2)} \\
&= F_{n+1} + F_n \\
&= F_{n+2}
\end{aligned}$$

and hence the result,

$$\det[a_{ij}]_n = F_{n+1}.$$

**25.** [*M21*] Show that

$$2^n F_n = 2 \sum_{k \text{ odd}} \binom{n}{k} 5^{(k-1)/2}.$$

**Proposition.** $2^n F_n = 2 \sum_{k\ odd} \binom{n}{k} 5^{(k-1)/2}$.

*Proof.* Let $n$ be an arbitrary, nonnegative integer. We must show that

$$2^n F_n = 2 \sum_{k \text{ odd}} \binom{n}{k} 5^{(k-1)/2}.$$

But

$$\begin{aligned} F_n &= \frac{1}{\sqrt{5}} \left(\phi^n - \hat{\phi}^n\right) \\ \iff \quad \sqrt{5} F_n &= \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \\ \iff \quad 2^n \sqrt{5} F_n &= \left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n \\ \iff \quad 2^n F_n &= \frac{1}{\sqrt{5}} \left(\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n\right). \end{aligned}$$

Therefore

$$\begin{aligned} 2^n F_n &= \frac{1}{\sqrt{5}} \left(\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n\right) \\ &= \frac{1}{\sqrt{5}} \left(\sum_k \binom{n}{k} 5^{k/2} - \sum_k \binom{n}{k} (-1)^k 5^{k/2}\right) \\ &= \sum_k \binom{n}{k} 5^{(k-1)/2} - \sum_k \binom{n}{k} (-1)^k 5^{(k-1)/2} \\ &= \sum_{k \text{ even}} \binom{n}{k} 5^{(k-1)/2} + \sum_{k \text{ odd}} \binom{n}{k} 5^{(k-1)/2} \\ &\quad - \sum_{k \text{ even}} \binom{n}{k} (-1)^k 5^{(k-1)/2} - \sum_{k \text{ odd}} \binom{n}{k} (-1)^k 5^{(k-1)/2} \\ &= \sum_{k \text{ even}} \binom{n}{k} 5^{(k-1)/2} + \sum_{k \text{ odd}} \binom{n}{k} 5^{(k-1)/2} \\ &\quad - \sum_{k \text{ even}} \binom{n}{k} 5^{(k-1)/2} + \sum_{k \text{ odd}} \binom{n}{k} 5^{(k-1)/2} \\ &= \sum_{k \text{ odd}} \binom{n}{k} 5^{(k-1)/2} + \sum_{k \text{ odd}} \binom{n}{k} 5^{(k-1)/2} \\ &= 2 \sum_{k \text{ odd}} \binom{n}{k} 5^{(k-1)/2} \end{aligned}$$

as we needed to show. □

▶ **26.** [*M20*] Using the previous exercise, show that $F_p \equiv 5^{(p-1)/2}$ (modulo $p$) if $p$ is an odd prime.

**Proposition.** $F_p \equiv 5^{(p-1)/2} \pmod{p}$ *if $p$ is an odd prime.*

*Proof.* Let $p$ be an arbitrary odd prime so that $p > 2$. We must show that

$$F_p \equiv 5^{(p-1)/2} \pmod{p}.$$

By *Fermat's theorem*, Theorem 1.2.4-F,

$$2^p \equiv 2 \pmod{p} \quad \iff \quad 1 \equiv 2^{p-1} \pmod{p}.$$

Then, by exercise 25,

$$2^p F_p = 2 \sum_{k \text{ odd}} \binom{p}{k} 5^{(k-1)/2}.$$

And so

$$\begin{aligned} & 2^p F_p \equiv 2^{p-1} 2 \sum_{k \text{ odd}} \binom{p}{k} 5^{(k-1)/2} \pmod{p} \\ \iff \quad & 2^p F_p \equiv 2^p \sum_{k \text{ odd}} \binom{p}{k} 5^{(k-1)/2} \pmod{p} \\ \iff \quad & F_p \equiv \sum_{k \text{ odd}} \binom{p}{k} 5^{(k-1)/2} \pmod{p}. \end{aligned}$$

Then

$$\begin{aligned} F_p &\equiv \sum_{k \text{ odd}} \binom{p}{k} 5^{(k-1)/2} \\ &\equiv \binom{p}{p} 5^{(p-1)/2} + \sum_{\substack{1 \le k \le p-1 \\ k \text{ odd}}} \binom{p}{k} 5^{(k-1)/2} \\ &\equiv 5^{(p-1)/2} + \sum_{\substack{1 \le k \le p-1 \\ k \text{ odd}}} \binom{p}{k} 5^{(k-1)/2} \\ &\equiv 5^{(p-1)/2} + 0 && \text{by exercise 1.2.6-10(b)} \\ &\equiv 5^{(p-1)/2} \pmod{p} \end{aligned}$$

as we needed to show. □

**27.** [*M20*] Using the previous exercise, show that if $p$ is a prime different from 5, then either $F_{p-1}$ or $F_{p+1}$ (not both) is a multiple of $p$.

**Proposition.** *If $p$ is a prime different from 5, then either $p \mid F_{p-1}$ or $p \mid F_{p+1}$ (exclusively).*

*Proof.* Let $p$ be an arbitrary prime different from 5. We must show that

$$p \mid F_{p-1} \quad \text{or} \quad p \mid F_{p+1}$$

(exclusively). In the case that $p = 2$,

$$2 \mid F_{2+1} \quad \text{and} \quad 2 \nmid F_{2-1}$$

since $k2 = F_{2+1} = F_3 = 2$ for $k = 1$ but $2 > 1 = F_1 = F_{2-1}$. Hereafter, we consider the case that $p$ is an odd prime different from 5. By Eq. (4),

$$\begin{aligned} & F_{p+1}F_{p-1} - F_p^2 = (-1)^p \\ \iff \quad & F_{p+1}F_{p-1} - F_p^2 = -1 \\ \iff \quad & F_{p+1}F_{p-1} = F_p^2 - 1 \\ \implies \quad & F_{p+1}F_{p-1} \equiv F_p^2 - 1 \pmod{p}. \end{aligned}$$

From the previous exercise,

$$\begin{aligned} &F_p \equiv 5^{(p-1)/2} \pmod{p} \\ \iff\quad &F_p^2 \equiv 5^{p-1} \pmod{p} \\ \iff\quad &F_p^2 - 1 \equiv 5^{p-1} - 1 \pmod{p}; \end{aligned}$$

and by *Fermat's theorem*, Theorem 1.2.4-F,

$$\begin{aligned} &5^p \equiv 5 \pmod{p} \\ \iff\quad &5^{p-1} \equiv 1 \pmod{p} \\ \iff\quad &5^{p-1} - 1 \equiv 0 \pmod{p}. \end{aligned}$$

And so,

$$\begin{aligned} F_{p+1}F_{p-1} &\equiv F_p^2 - 1 \\ &\equiv 5^{p-1} - 1 \\ &\equiv 0 \pmod{p}. \end{aligned}$$

That is, that

$$p \mid F_{p-1} \quad \text{or} \quad p \mid F_{p+1}.$$

To see that this is exclusive, consider the case that $p \mid F_{p-1}$. Then $F_{p-1} = mp$ for some $m$. If we assume $F_{p+1} = np$ for some $n$,

$$F_{p+1} = F_p + F_{p-1} = F_p + mp = np,$$

then $p \mid F_p$. But by *Fermat's theorem*, again,

$$\begin{aligned} &5^p \equiv 5 \pmod{p} \\ \iff\quad &5^{p-1} \equiv 1 \pmod{p} \\ \iff\quad &5^{(p-1)/2} \equiv 1 \pmod{p}, \end{aligned}$$

and from the previous exercise

$$\begin{aligned} F_p &\equiv 5^{(p-1)/2} \\ &\equiv 1 \pmod{p}, \end{aligned}$$

contradicting the assumption $F_{p+1} = np$, so that $p \nmid F_{p+1}$ if $p \mid F_{p-1}$. Similarly, consider the case that $p \mid F_{p+1}$. Then $F_{p+1} = np$ for some $n$. If we assume $F_{p-1} = mp$ for some $m$,

$$F_{p-1} = -F_p + F_{p+1} = -F_p + mp = np,$$

then $p \mid F_p$. But as with the previous case,

$$F_p \equiv 1 \pmod{p}$$

contradicting the assumption $F_{p-1} = mp$, so that $p \nmid F_{p-1}$ if $p \mid F_{p+1}$. This is what we needed to show. □

**28.** [*M21*] What is $F_{n+1} - \phi F_n$?

We have

$$\begin{aligned}
F_{n+1} - \phi F_n &= \frac{1}{\sqrt{5}}\left(\phi^{n+1} - \hat{\phi}^{n+1}\right) - \phi\frac{1}{\sqrt{5}}\left(\phi^n - \hat{\phi}^n\right) \\
&= \frac{1}{\sqrt{5}}\left(\phi^{n+1} - \hat{\phi}^{n+1} - \phi\phi^n + \phi\hat{\phi}^n\right) \\
&= \frac{1}{\sqrt{5}}\left(\phi^{n+1} - \phi^{n+1} + \hat{\phi}^n\phi - \hat{\phi}^n\hat{\phi}\right) \\
&= \frac{1}{\sqrt{5}}\left(0 + \hat{\phi}^n\left(\phi - \hat{\phi}\right)\right) \\
&= \hat{\phi}^n\frac{\phi - \hat{\phi}}{\sqrt{5}} \\
&= \hat{\phi}^n\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)\frac{1}{\sqrt{5}} \\
&= \hat{\phi}^n\left(\frac{1}{2} + \frac{\sqrt{5}}{2} - \frac{1}{2} + \frac{\sqrt{5}}{2}\right)\frac{1}{\sqrt{5}} \\
&= \hat{\phi}^n\left(\frac{\sqrt{5}}{2} + \frac{\sqrt{5}}{2}\right)\frac{1}{\sqrt{5}} \\
&= \hat{\phi}^n\frac{2\sqrt{5}}{2}\frac{1}{\sqrt{5}} \\
&= \hat{\phi}^n\frac{\sqrt{5}}{\sqrt{5}} \\
&= \hat{\phi}^n.
\end{aligned}$$

▶ **29.** [*M23*] (*Fibonomial coefficients.*) Édouard Lucas defined the quantities

$$\binom{n}{k}_{\mathcal{F}} = \frac{F_nF_{n-1}\dots F_{n-k+1}}{F_kF_{k-1}\dots F_1} = \prod_{j=1}^{k}\left(\frac{F_{n-k+j}}{F_j}\right)$$

in a manner analogous to binomial coefficients. (a) Make a table of $\binom{n}{k}_{\mathcal{F}}$ for $0 \le k \le n \le 6$. (b) Show that $\binom{n}{k}_{\mathcal{F}}$ is always an integer because we have

$$\binom{n}{k}_{\mathcal{F}} = F_{k-1}\binom{n-1}{k}_{\mathcal{F}} + F_{n-k+1}\binom{n-1}{k-1}_{\mathcal{F}}.$$

a) The table of *Fibonomial coefficients* $\binom{n}{k}_{\mathcal{F}}$ for $0 \le k \le n \le 6$ would appear as below.

| $n$ | $\binom{n}{0}_{\mathcal{F}}$ | $\binom{n}{1}_{\mathcal{F}}$ | $\binom{n}{2}_{\mathcal{F}}$ | $\binom{n}{3}_{\mathcal{F}}$ | $\binom{n}{4}_{\mathcal{F}}$ | $\binom{n}{5}_{\mathcal{F}}$ | $\binom{n}{6}_{\mathcal{F}}$ |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 3 | 1 | 2 | 2 | 1 | 0 | 0 | 0 |
| 4 | 1 | 3 | 6 | 3 | 1 | 0 | 0 |
| 5 | 1 | 5 | 15 | 15 | 5 | 1 | 0 |
| 6 | 1 | 8 | 40 | 60 | 40 | 8 | 1 |

This is based on the observations that

$$\begin{aligned}
\binom{n}{0}_{\mathcal{F}} &= \prod_{1\le j\le 0} \frac{F_{n-0+j}}{F_j} = 1,\\
\binom{n}{1}_{\mathcal{F}} &= \prod_{1\le j\le 1} \frac{F_{n-1+j}}{F_j} = \frac{F_n}{F_1} = F_n,\\
\binom{n}{2}_{\mathcal{F}} &= \prod_{1\le j\le 2} \frac{F_{n-2+j}}{F_j} = \frac{F_{n-1}}{F_1}\frac{F_n}{F_2} = F_{n-1}F_n,\\
\binom{n}{3}_{\mathcal{F}} &= \prod_{1\le j\le 3} \frac{F_{n-3+j}}{F_j} = \frac{F_{n-2}}{F_1}\frac{F_{n-1}}{F_2}\frac{F_n}{F_3} = \frac{1}{2}F_{n-2}F_{n-1}F_n,\\
\binom{n}{4}_{\mathcal{F}} &= \prod_{1\le j\le 4} \frac{F_{n-4+j}}{F_j} = \frac{F_{n-3}}{F_1}\frac{F_{n-2}}{F_2}\frac{F_{n-1}}{F_3}\frac{F_n}{F_4} = \frac{1}{6}F_{n-3}F_{n-2}F_{n-1}F_n,\\
\binom{n}{5}_{\mathcal{F}} &= \prod_{1\le j\le 5} \frac{F_{n-5+j}}{F_j} = \frac{F_{n-4}}{F_1}\frac{F_{n-3}}{F_2}\frac{F_{n-2}}{F_3}\frac{F_{n-1}}{F_4}\frac{F_n}{F_5}\\
&= \frac{1}{30}F_{n-4}F_{n-3}F_{n-2}F_{n-1}F_n, \text{ and}\\
\binom{n}{6}_{\mathcal{F}} &= \prod_{1\le j\le 6} \frac{F_{n-6+j}}{F_j} = \frac{F_{n-5}}{F_1}\frac{F_{n-4}}{F_2}\frac{F_{n-3}}{F_3}\frac{F_{n-2}}{F_4}\frac{F_{n-1}}{F_5}\frac{F_n}{F_6}\\
&= \frac{1}{240}F_{n-5}F_{n-4}F_{n-3}F_{n-2}F_{n-1}F_n.
\end{aligned}$$

b) We may show that $\binom{n}{k}_{\mathcal{F}}$ is always an integer by proving the recursive relation below.

**Proposition.** $\binom{n}{k}_{\mathcal{F}} = F_{k-1}\binom{n-1}{k}_{\mathcal{F}} + F_{n-k+1}\binom{n-1}{k-1}_{\mathcal{F}}$.

*Proof.* We must show that

$$\binom{n}{k}_{\mathcal{F}} = F_{k-1}\binom{n-1}{k}_{\mathcal{F}} + F_{n-k+1}\binom{n-1}{k-1}_{\mathcal{F}}$$

for $1 \le k \le n$. But

$$\begin{aligned}
&\binom{n}{k}_{\mathcal{F}} \\
&= \prod_{1\le j\le k} \frac{F_{n-k+j}}{F_j} \\
&= \frac{F_k F_{n-k+1} + F_{k-1}F_{n-k}}{F_{n-k+k}} \prod_{1\le j\le k} \frac{F_{n-k+j}}{F_j} && \text{by Eq. (6)} \\
&= \frac{F_{n-k}F_{k-1} + F_{n-k+1}F_k}{F_n} \prod_{1\le j\le k} \frac{F_{n-k+j}}{F_j} \\
&= \frac{F_{n-k}F_{k-1}}{F_n} \prod_{1\le j\le k} \frac{F_{n-k+j}}{F_j} + \frac{F_{n-k+1}F_k}{F_n} \prod_{1\le j\le k} \frac{F_{n-k+j}}{F_j} \\
&= F_{n-k}\frac{F_{k-1}}{F_n} \prod_{1\le j\le k} \frac{F_{n-k+j}}{F_j} + F_{n-k+1}\frac{F_k}{F_n} \prod_{1\le j\le k} \frac{F_{n-k+j}}{F_j} \\
&= F_{n-k}\frac{F_{k-1}}{F_n} \prod_{2\le j\le k+1} \frac{F_{n-k+j-1}}{F_{j-1}} + F_{n-k+1} \prod_{1\le j\le k-1} \frac{F_{n-k+j}}{F_j} \\
&= F_{n-k}\frac{F_{k-1}}{F_n} \frac{F_n \prod_{2\le j\le k} F_{n-k+j-1}}{\prod_{1\le j\le k} F_j} + F_{n-k+1} \prod_{1\le j\le k-1} \frac{F_{n-k+j}}{F_j} \\
&= F_{n-k}F_{k-1} \frac{\prod_{2\le j\le k} F_{n-k+j-1}}{\prod_{1\le j\le k} F_j} + F_{n-k+1} \prod_{1\le j\le k-1} \frac{F_{n-k+j}}{F_j} \\
&= F_{k-1} \frac{F_{n-k+1-1} \prod_{2\le j\le k} F_{n-k+j-1}}{\prod_{1\le j\le k} F_j} + F_{n-k+1} \prod_{1\le j\le k-1} \frac{F_{n-k+j}}{F_j} \\
&= F_{k-1} \frac{\prod_{1\le j\le k} F_{n-k+j-1}}{\prod_{1\le j\le k} F_j} + F_{n-k+1} \prod_{1\le j\le k-1} \frac{F_{n-k+j}}{F_j} \\
&= F_{k-1} \prod_{1\le j\le k} \frac{F_{n-k+j-1}}{F_j} + F_{n-k+1} \prod_{1\le j\le k-1} \frac{F_{n-k+j}}{F_j} \\
&= F_{k-1} \prod_{1\le j\le k} \frac{F_{n-1-k+j}}{F_j} + F_{n-k+1} \prod_{1\le j\le k-1} \frac{F_{n-1-(k-1)+j}}{F_j} \\
&= F_{k-1}\binom{n-1}{k}_{\mathcal{F}} + F_{n-k+1}\binom{n-1}{k-1}_{\mathcal{F}}
\end{aligned}$$

as we needed to show. □

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[É. Lucas, *Amer. J. Math.* **1** (1878), 201–204]

► **30.** [*M38*] (D. Jarden, T. Motzkin.) The sequence of $m$th powers of Fibonacci numbers satisfies a recurrence relation in which each term depends on the preceding $m+1$ terms. Show that

$$\sum_k \binom{m}{k}_{\mathcal{F}} (-1)^{\lceil (m-k)/2 \rceil} F_{n+k}^{m-1} = 0, \qquad \text{if } m > 0.$$

For example, when $m = 3$ we get the identity $F_n^2 - 2F_{n+1}^2 - 2F_{n+2}^2 + F_{n+3}^2 = 0$.

We may prove the equality as a particular case of the proof outlined by Cooper and Kennedy.[1]

[1] Curtis Cooper, and Robert E. Kennedy, Proof of a Result by Jarden by Generalizing a Proof by Carlitz, *Fibonacci Quarterly* **33** (1995) 304-311.

**Proposition.** $\sum_k \binom{m}{k}_{\mathcal{F}}(-1)^{\lceil (m-k)/2\rceil} F_{n+k}^{m-1} = 0$ *if* $m > 0$.

*Proof.* Let $m, n, k$ be arbitrary integers such that $m > 0$. We must show that

$$\sum_k \binom{m}{k}_{\mathcal{F}}(-1)^{\lceil (m-k)/2\rceil} F_{n+k}^{m-1} = 0,$$

or equivalently for $n' = n + m$ that

$$\begin{aligned}
&\sum_k \binom{m}{k}_{\mathcal{F}}(-1)^{\lceil (m-k)/2\rceil} F_{n+k}^{m-1} = 0 \\
\iff\quad &\sum_{0\le k\le m} \binom{m}{k}_{\mathcal{F}}(-1)^{\lceil (m-k)/2\rceil} F_{n+k}^{m-1} = 0 \\
\iff\quad &\sum_{0\le k\le m} \binom{m}{m-k}_{\mathcal{F}}(-1)^{\lceil (m-k)/2\rceil} F_{n'-(m-k)}^{m-1} = 0 \\
\iff\quad &\sum_{0\le m-k\le m} \binom{m}{m-k}_{\mathcal{F}}(-1)^{\lceil (m-k)/2\rceil} F_{n'-(m-k)}^{m-1} = 0 \\
\iff\quad &\sum_{0\le k\le m} \binom{m}{k}_{\mathcal{F}}(-1)^{\lceil k/2\rceil} F_{n'-k}^{m-1} = 0.
\end{aligned}$$

*Preliminary result 30.1.* From Eq. (14), we have that

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \hat{\phi}^n\right) = \frac{\phi^n - \hat{\phi}^n}{\phi - \hat{\phi}}. \tag{30.1}$$

*Preliminary result 30.2.* As shown in exercise 14, we may show that

$$F_{n+1} = \sum_{0\le k\le n} \binom{k}{n-k}. \tag{30.2}$$

In the case that $n = 0$,

$$F_1 = 1 = 1 = \binom{0}{0} = \sum_{0\le k\le 0} \binom{k}{0-k};$$

and in the case that $n = 1$,

$$F_2 = 1 = 0 + 1 = \binom{0}{1} + \binom{1}{1} = \sum_{0\le k\le 1} \binom{k}{1-k};$$

Then assuming

$$F_{n+1} = \sum_{0\le k\le n} \binom{k}{n-k},$$

we must show

$$F_{n+2} = \sum_{0\le k\le n+1} \binom{k}{n-k+1}.$$

But

$$\begin{aligned}
F_{n+2} &= F_{n+1} + F_n \\
&= \sum_{0 \le k \le n} \binom{k}{n-k} + \sum_{0 \le k \le n-1} \binom{k}{n-k-1} \\
&= \binom{n}{0} + \sum_{0 \le k \le n-1} \binom{k}{n-k} + \sum_{0 \le k \le n-1} \binom{k}{n-k-1} \\
&= 1 + \sum_{0 \le k \le n-1} \left( \binom{k}{n-k} + \binom{k}{n-k-1} \right) \\
&= 1 + \sum_{0 \le k \le n-1} \binom{k+1}{n-k} \\
&= 1 + \sum_{1 \le k \le n} \binom{k}{n-k+1} \\
&= \binom{n+1}{0} + \sum_{1 \le k \le n} \binom{k}{n-k+1} \\
&= \binom{n+1}{n-(n+1)+1} + \sum_{1 \le k \le n} \binom{k}{n-k+1} \\
&= \sum_{1 \le k \le n+1} \binom{k}{n-k+1} \\
&= 0 + \sum_{1 \le k \le n+1} \binom{k}{n-k+1} \\
&= \binom{0}{n-0+1} + \sum_{1 \le k \le n+1} \binom{k}{n-k+1} \\
&= \sum_{0 \le k \le n+1} \binom{k}{n-k+1}
\end{aligned}$$

and hence the result.

*Preliminary result 30.3.* We have that

$$F_n + F_{n-2} = \phi^{n-1} + \hat{\phi}^{n-1} \tag{30.3}$$

since by definition and (30.1)

$$\begin{aligned}
F_n + F_{n-2} &= F_n + F_n - F_{n-1} \\
&= 2F_n - F_{n-1} \\
&= 2\frac{\phi^n - \hat{\phi}^n}{\phi - \hat{\phi}} - \frac{\phi^{n-1} - \hat{\phi}^{n-1}}{\phi - \hat{\phi}} \\
&= \frac{2\phi^n - 2\hat{\phi}^n - \phi^{n-1} + \hat{\phi}^{n-1}}{\phi - \hat{\phi}} \\
&= \frac{2\phi^n - \phi^{n-1} + \hat{\phi}^{n-1} - 2\hat{\phi}^n}{\phi - \hat{\phi}} \\
&= \frac{2\phi\phi^{n-1} - \phi^{n-1} + \hat{\phi}^{n-1} - 2\hat{\phi}\hat{\phi}^{n-1}}{\phi - \hat{\phi}} \\
&= \frac{(2\phi - 1)\,\phi^{n-1} + \left(1 - 2\hat{\phi}\right)\hat{\phi}^{n-1}}{\phi - \hat{\phi}} \\
&= \frac{(2\phi - 1)\,\phi^{n-1} + (2\phi - 1)\,\hat{\phi}^{n-1}}{\phi - \hat{\phi}} \\
&= \frac{(2\phi - 1)\left(\phi^{n-1} + \hat{\phi}^{n-1}\right)}{\phi - \hat{\phi}} \\
&= \frac{\left(\phi - \hat{\phi}\right)\left(\phi^{n-1} + \hat{\phi}^{n-1}\right)}{\phi - \hat{\phi}} \\
&= \phi^{n-1} + \hat{\phi}^{n-1}.
\end{aligned}$$

*Preliminary result 30.4.* We have that

$$F_n F_{n-2} - F_{n-1}^2 = \phi^{n-1}\hat{\phi}^{n-1} \tag{30.4}$$

since by definition and (30.1)

$$\begin{aligned}
F_nF_{n-2} - F_{n-1}^2 &= F_n\left(F_n - F_{n-1}\right) - F_{n-1}^2 \\
&= F_n^2 - F_nF_{n-1} - F_{n-1}^2 \\
&= F_n^2 - F_{n-1}\left(F_n + F_{n-1}\right) \\
&= F_n^2 - F_{n-1}F_{n+1} \\
&= \left(\frac{\phi^n - \hat{\phi}^n}{\phi - \hat{\phi}}\right)^2 - \frac{\phi^{n-1} - \hat{\phi}^{n-1}}{\phi - \hat{\phi}}\frac{\phi^{n+1} - \hat{\phi}^{n+1}}{\phi - \hat{\phi}} \\
&= \frac{\phi^{2n} - 2\phi^n\hat{\phi}^n + \hat{\phi}^{2n} - \phi^{2n} + \phi^{n+1}\hat{\phi}^{n-1} + \phi^{n-1}\hat{\phi}^{n+1} - \hat{\phi}^{2n}}{\left(\phi - \hat{\phi}\right)^2} \\
&= \frac{\phi^{n+1}\hat{\phi}^{n-1} - 2\phi^n\hat{\phi}^n + \phi^{n-1}\hat{\phi}^{n+1}}{\left(\phi - \hat{\phi}\right)^2} \\
&= \frac{\left(\phi^{n-1}\hat{\phi}^{n-1}\right)\left(\phi^2 - 2\phi\hat{\phi} + \hat{\phi}^2\right)}{\left(\phi - \hat{\phi}\right)^2} \\
&= \frac{\left(\phi^{n-1}\hat{\phi}^{n-1}\right)\left(\phi - \hat{\phi}\right)^2}{\left(\phi - \hat{\phi}\right)^2} \\
&= \phi^{n-1}\hat{\phi}^{n-1}.
\end{aligned}$$

*Preliminary result 30.5.* We may show that

$$\left(F_kx + F_{k-1}\right)^r\left(F_{k+1}x + F_k\right)^{n-r} = \sum_{s_1,\cdots,s_k}\binom{n-r}{s_1}\binom{n-s_1}{s_2}\cdots\binom{n-s_{k-1}}{s_k}x^{n-s_k} \tag{30.5}$$

for $0 \le r \le n$. In the case that $k = 1$,

$$\begin{aligned}
&\left(F_1x + F_{1-1}\right)^r\left(F_{1+1}x + F_1\right)^{n-r} \\
&\quad = x^r\left(x+1\right)^{n-r} \\
&\quad = x^r\left(1+x\right)^{n-r} \\
&\quad = x^r\sum_{0\le s_1\le n-r}\binom{n-r}{s_1}x^{s_1} \\
&\quad = \sum_{0\le n-r-s_1\le n-r}\binom{n-r}{n-r-s_1}x^{n-r-s_1}x^r \\
&\quad = \sum_{0\le s_1\le n-r}\binom{n-r}{s_1}x^{n-s_1} \\
&\quad = \sum_{s_1}\binom{n-r}{s_1}x^{n-s_1}.
\end{aligned}$$

Then, assuming

$$\begin{aligned}
&\left(F_kx + F_{k-1}\right)^r\left(F_{k+1}x + F_k\right)^{n-r} \\
&\quad = \sum_{s_1,\cdots,s_k}\binom{n-r}{s_1}\binom{n-s_1}{s_2}\cdots\binom{n-s_{k-1}}{s_k}x^{n-s_k},
\end{aligned}$$

we must show that

$$\begin{aligned}&(F_{k+1}x+F_k)^r\,(F_{k+2}x+F_{k+1})^{n-r}\\&\quad=\sum_{s_1,\cdots,s_{k+1}}\binom{n-r}{s_1}\binom{n-s_1}{s_2}\cdots\binom{n-s_k}{s_{k+1}}x^{n-s_{k+1}}.\end{aligned}$$

But for $x'=1+x^{-1}$,

$$\begin{aligned}&(F_{k+1}x+F_k)^r\,(F_{k+2}x+F_{k+1})^{n-r}\\
&=x^r\left(F_{k+1}+F_kx^{-1}\right)^r x^{n-r}\left(F_{k+2}+F_{k+1}x^{-1}\right)^{n-r}\\
&=x^n\left(F_k+F_{k-1}+F_kx^{-1}\right)^r\left(F_{k+1}+F_k+F_{k+1}x^{-1}\right)^{n-r}\\
&=x^n\left(F_k+F_kx^{-1}+F_{k-1}\right)^r\left(F_{k+1}+F_{k+1}x^{-1}+F_k\right)^{n-r}\\
&=x^n\left(F_k\left(1+x^{-1}\right)+F_{k-1}\right)^r\left(F_{k+1}\left(1+x^{-1}\right)+F_k\right)^{n-r}\\
&=x^n\left(F_kx'+F_{k-1}\right)^r\left(F_{k+1}x'+F_k\right)^{n-r}\\
&=x^n\sum_{s_1,\cdots,s_k}\binom{n-r}{s_1}\binom{n-s_1}{s_2}\cdots\binom{n-s_{k-1}}{s_k}x'^{n-s_k}\\
&=\sum_{s_1,\cdots,s_k}\binom{n-r}{s_1}\binom{n-s_1}{s_2}\cdots\binom{n-s_{k-1}}{s_k}\left(1+x^{-1}\right)^{n-s_k}x^n\\
&=\sum_{s_1,\cdots,s_k}\binom{n-r}{s_1}\binom{n-s_1}{s_2}\cdots\binom{n-s_{k-1}}{s_k}(x+1)^{n-s_k}x^{s_k}\\
&=\sum_{s_1,\cdots,s_k}\binom{n-r}{s_1}\binom{n-s_1}{s_2}\cdots\binom{n-s_{k-1}}{s_k}\sum_{s_{k+1}}\binom{n-s_k}{s_{k+1}}x^{n-s_k-s_{k+1}}x^{s_k}\\
&=\sum_{s_1,\cdots,s_{k+1}}\binom{n-r}{s_1}\binom{n-s_1}{s_2}\cdots\binom{n-s_k}{s_{k+1}}x^{n-s_{k+1}}\end{aligned}$$

and hence the result.

*Preliminary Result 30.6.* From Eq. (1.2.6-19), we have that

$$\binom{n}{k}=(-1)^{n-k}\binom{-k-1}{n-k}. \tag{30.6}$$

*Preliminary Result 30.7.* Define

$$\mathbf{A}_{n+1}=[a_{ij}]_{n+1}=\left[\binom{i}{n-j}\right]_{n+1}=\begin{bmatrix}\binom{0}{n}&\binom{0}{n-1}&\cdots&\binom{0}{0}\\\binom{1}{n}&\binom{1}{n-1}&\cdots&\binom{1}{0}\\\vdots&\vdots&\cdots&\vdots\\\binom{n}{n}&\binom{n}{n-1}&\cdots&\binom{n}{0}\end{bmatrix}_{n+1}.$$

Then

$$\operatorname{tr}\left(\mathbf{A}_{n+1}^k\right)=\frac{F_{kn+k}}{F_k} \tag{30.7}$$

for $k>0$, where $\operatorname{tr}(\mathbf{B}_{n+1})$ is the *trace* of $\mathbf{B}_{n+1}$ defined as

$$\operatorname{tr}(\mathbf{B}_{n+1})=\sum_{0\le i\le n}b_{ij}.$$

Note that the case $k = 1$ is (30.2). But by (30.5),

$$\begin{aligned}
&(F_k x + F_{k-1})^r (F_{k+1} x + F_k)^{n-r} \\
&\quad = \sum_{s_1,\cdots,s_k} \binom{n-r}{s_1}\binom{n-s_1}{s_2}\cdots\binom{n-s_{k-1}}{s_k} x^{n-s_k} \\
&\iff \quad (F_k x + F_{k-1})^r (F_{k+1} x + F_k)^{n-r} x^r \\
&\quad = \sum_{s_1,\cdots,s_k} \binom{n-r}{s_1}\binom{n-s_1}{s_2}\cdots\binom{n-s_{k-1}}{s_k} x^{n+r-s_k} \\
&\iff \quad \sum_{0\le r\le n} (F_k x + F_{k-1})^r (F_{k+1} x + F_k)^{n-r} x^r \\
&\quad = \sum_{0\le r\le n}\sum_{s_1,\cdots,s_k} \binom{n-r}{s_1}\binom{n-s_1}{s_2}\cdots\binom{n-s_{k-1}}{s_k} x^{n+r-s_k} \\
&\iff \quad \sum_{n\ge 0}\sum_{0\le r\le n} (F_k x + F_{k-1})^r (F_{k+1} x + F_k)^{n-r} x^r \\
&\quad = \sum_{n\ge 0}\sum_{0\le r\le n}\sum_{s_1,\cdots,s_k} \binom{n-r}{s_1}\binom{n-s_1}{s_2}\cdots\binom{n-s_{k-1}}{s_k} x^{r-s_k} x^n
\end{aligned}$$

and since

$$\begin{aligned}
\operatorname{tr}\left(\mathbf{A}_{n+1}^k\right) &= \sum_{0\le i\le n} \sum_{\substack{s_1,s_2,\cdots,s_k \\ i=s_1=s_2=\cdots=s_k}} \binom{i}{n-s_1}\binom{s_1}{n-s_2}\cdots\binom{s_{k-1}}{n-s_k} \\
&= \sum_{0\le n-i\le n} \sum_{\substack{n-s_1,n-s_2,\cdots,n-s_k \\ n-i=n-s_1=n-s_2=\cdots=n-s_k}} \binom{i}{n-s_1}\binom{s_1}{n-s_2}\cdots\binom{s_{k-1}}{n-s_k} \\
&= \sum_{0\le r\le n} \sum_{\substack{s_1,s_2,\cdots,s_k \\ i=s_1=s_2=\cdots=s_k}} \binom{n-r}{s_1}\binom{n-s_1}{s_2}\cdots\binom{n-s_{k-1}}{s_k} \\
&= \sum_{0\le r\le n} \sum_{\substack{s_1,s_2,\cdots,s_k \\ i=s_1=s_2=\cdots=s_k}} \binom{n-r}{s_1}\binom{n-s_1}{s_2}\cdots\binom{n-s_{k-1}}{s_k} x^0 \\
&= \sum_{0\le r\le n} \sum_{\substack{s_1,s_2,\cdots,s_k \\ i=s_1=s_2=\cdots=s_k}} \binom{n-r}{s_1}\binom{n-s_1}{s_2}\cdots\binom{n-s_{k-1}}{s_k} x^{r-s_k}
\end{aligned}$$

we have that

$$\begin{aligned}
&\sum_{n\ge 0}\sum_{0\le r\le n}\sum_{s_1,\cdots,s_k} \binom{n-r}{s_1}\binom{n-s_1}{s_2}\cdots\binom{n-s_{k-1}}{s_k} x^{r-s_k} x^n \\
&\quad = \sum_{n\ge 0} \operatorname{tr}\left(\mathbf{A}_{n+1}^k\right) x^n \\
&\quad = \sum_{n\ge 0}\sum_{0\le r\le n} (F_k x + F_{k-1})^r (F_{k+1} x + F_k)^{n-r} x^r.
\end{aligned}$$

But

$$
\begin{aligned}
&\sum_{0\le r\le n} (F_kx+F_{k-1})^r\,(F_{k+1}x+F_k)^{n-r}\,x^r\\
&\quad=\sum_{0\le r\le n}\sum_{0\le s\le r}\binom{r}{s}(F_kx)^s\,F_{k-1}^{r-s}\sum_{0\le t\le n-r}\binom{n-r}{t}(F_{k+1}x)^t\,F_k^{n-r-t}x^r\\
&\quad=\sum_{0\le r\le n}\sum_{0\le s\le r}\sum_{0\le t\le n-r}\binom{r}{s}\binom{n-r}{t}F_{k+1}^tF_k^sF_k^{n-r-t}F_{k-1}^{r-s}x^sx^tx^r\\
&\quad=\sum_{0\le r\le n}\sum_{0\le s\le r}\sum_{0\le t\le n-r}\binom{r}{s}\binom{n-r}{t}F_{k+1}^tF_k^{n-r+s-t}F_{k-1}^{r-s}x^{r+s+t}\\
&\quad=\sum_{n=r+s+t}\binom{r}{s}\binom{n-r}{t}F_{k+1}^tF_k^{n-r+s-t}F_{k-1}^{r-s}x^n\\
&\quad=\sum_{n=r+s+t}\binom{r}{s}\binom{n-r}{n-r-s}F_{k+1}^{n-r-s}F_k^{2s}F_{k-1}^{r-s}x^n\\
&\quad=\sum_{n\ge r+s}\binom{r}{s}\binom{n-r}{s}F_{k+1}^{n-r-s}F_k^{2s}F_{k-1}^{r-s}x^n\\
&\quad=\sum_{r,s\ge 0}\binom{r}{s}F_{k-1}^{r-s}F_k^{2s}x^{r+s}\sum_{n\ge r+s}\binom{n-r}{s}(F_{k+1}x)^{n-r-s}\\
&\quad=\sum_{r,s\ge 0}\binom{r}{s}F_{k-1}^{r-s}F_k^{2s}x^{r+s}\sum_{n\ge r+s}\binom{n-r}{s}F_{k+1}^{n-r-s}x^{n-r-s}\\
&\quad=\sum_{r,s\ge 0}\binom{r}{s}F_{k-1}^{r-s}F_k^{2s}x^{r+s}\sum_{s\le n-r}(-1)^{n-r-s}\binom{-s-1}{n-r-s}(F_{k+1}x)^{n-r-s}\qquad\text{by (30.6)}\\
&\quad=\sum_{r,s\ge 0}\binom{r}{s}F_{k-1}^{r-s}F_k^{2s}x^{r+s}\sum_{0\le n-r-s}\binom{-s-1}{n-r-s}(-F_{k+1}x)^{n-r-s}\\
&\quad=\sum_{r,s\ge 0}\binom{r}{s}F_{k-1}^{r-s}F_k^{2s}x^{r+s}\,(1-F_{k+1}x)^{-s-1}\\
&\quad=\sum_{s\ge 0}F_k^{2s}x^{2s}\,(1-F_{k+1}x)^{-s-1}\sum_{r\ge 0}\binom{r}{s}F_{k-1}^{r-s}x^{r-s}\\
&\quad=\sum_{s\ge 0}F_k^{2s}x^{2s}\,(1-F_{k+1}x)^{-s-1}\sum_{s\le r}\binom{r}{s}F_{k-1}^{r-s}x^{r-s}\\
&\quad=\sum_{s\ge 0}F_k^{2s}x^{2s}\,(1-F_{k+1}x)^{-s-1}\sum_{s\le r}\binom{r}{s}(F_{k-1}x)^{r-s}\\
&\quad=\sum_{s\ge 0}F_k^{2s}x^{2s}\,(1-F_{k+1}x)^{-s-1}\sum_{s\le r}(-1)^{r-s}\binom{-s-1}{r-s}(F_{k-1}x)^{r-s}\qquad\text{by (30.6)}\\
&\quad=\sum_{s\ge 0}F_k^{2s}x^{2s}\,(1-F_{k+1}x)^{-s-1}\sum_{0\le r-s}\binom{-s-1}{r-s}(-F_{k-1}x)^{r-s}\\
&\quad=\sum_{s\ge 0}F_k^{2s}x^{2s}\,(1-F_{k+1}x)^{-s-1}\,(1-F_{k-1}x)^{-s-1}\\
&\quad=\frac{1}{(1-F_{k+1}x)\,(1-F_{k-1}x)}\sum_{s\ge 0}\left(\frac{F_k^2x^2}{(1-F_{k+1}x)\,(1-F_{k-1}x)}\right)^s.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{1}{(1-F_{k+1}x)(1-F_{k-1}x)}\sum_{s\geq 0}\left(\frac{F_k^2x^2}{(1-F_{k+1}x)(1-F_{k-1}x)}\right)^s\\
&=\frac{1}{(1-F_{k+1}x)(1-F_{k-1}x)}\frac{1}{1-\frac{F_k^2x^2}{(1-F_{k+1}x)(1-F_{k-1}x)}}\\
&=\frac{1}{(1-F_{k+1}x)(1-F_{k-1}x)}\frac{(1-F_{k+1}x)(1-F_{k-1}x)}{(1-F_{k+1}x)(1-F_{k-1}x)-F_k^2x^2}\\
&=\frac{1}{(1-F_{k+1}x)(1-F_{k-1}x)-F_k^2x^2}\\
&=\frac{1}{1-F_{k+1}x-F_{k-1}x+F_{k+1}F_{k-1}x^2-F_k^2x^2}\\
&=\frac{1}{1-(F_{k+1}+F_{k-1})\,x+(F_{k+1}F_{k-1}-F_k^2)\,x^2}\\
&=\frac{1}{1-\left(\phi^k+\hat{\phi}^k\right)x+(F_{k+1}F_{k-1}-F_k^2)\,x^2} && \text{by (30.3)}\\
&=\frac{1}{1-\left(\phi^k+\hat{\phi}^k\right)x+\phi^k\hat{\phi}^kx^2} && \text{by (30.4)}\\
&=\frac{1}{1-\phi^kx-\hat{\phi}^kx+\phi^k\hat{\phi}^kx^2}\\
&=\frac{1}{\phi^k-\hat{\phi}^k}\frac{\phi^k\left(1-\hat{\phi}^kx\right)-\hat{\phi}^k\left(1-\phi^kx\right)}{(1-\phi^kx)\left(1-\hat{\phi}^kx\right)}\\
&=\frac{1}{\phi^k-\hat{\phi}^k}\left(\frac{\phi^k}{1-\phi^kx}-\frac{\hat{\phi}^k}{1-\hat{\phi}^kx}\right)\\
&=\frac{1}{\phi^k-\hat{\phi}^k}\left(\phi^k\frac{1}{1-\phi^kx}-\hat{\phi}^k\frac{1}{1-\hat{\phi}^kx}\right)\\
&=\frac{1}{\phi^k-\hat{\phi}^k}\left(\phi^k\sum_{n\geq 0}\left(\phi^kx\right)^n-\hat{\phi}^k\sum_{n\geq 0}\left(\hat{\phi}^kx\right)^n\right)\\
&=\frac{1}{\phi^k-\hat{\phi}^k}\left(\phi^k\sum_{n\geq 0}\left(\phi^k\right)^nx^n-\hat{\phi}^k\sum_{n\geq 0}\left(\hat{\phi}^k\right)^nx^n\right)\\
&=\frac{1}{\phi^k-\hat{\phi}^k}\left(\sum_{n\geq 0}\phi^{kn+k}x^n-\sum_{n\geq 0}\hat{\phi}^{kn+k}x^n\right)\\
&=\frac{1}{\phi^k-\hat{\phi}^k}\sum_{n\geq 0}\left(\phi^{kn+k}-\hat{\phi}^{kn+k}\right)x^n\\
&=\sum_{n\geq 0}\frac{\phi^{kn+k}-\hat{\phi}^{kn+k}}{\phi^k-\hat{\phi}^k}x^n\\
&=\sum_{n\geq 0}\frac{\phi^{kn+k}-\hat{\phi}^{kn+k}}{\phi-\hat{\phi}}\frac{\phi-\hat{\phi}}{\phi^k-\hat{\phi}^k}x^n\\
&=\sum_{n\geq 0}\frac{F_{kn+k}}{F_k}x^n.
\end{aligned}
$$

And so,

$$\sum_{n\geq 0} \operatorname{tr}\left(\mathbf{A}_{n+1}^k\right) x^n = \sum_{n\geq 0} \frac{F_{kn+k}}{F_k} x^n,$$

hence the result.

*Preliminary Result 30.8.* We may show that the eigenvalues of $\mathbf{A}_{n+1}$ are

$$\lambda_j = \phi^j \hat{\phi}^{n-j} \tag{30.8}$$

for $0 \leq j \leq n$. Let

$$p_{\mathbf{A}_{n+1}}(x) = \det(x\mathbf{I}_{n+1} - \mathbf{A}_{n+1})$$

be the characteristic polynomial of $\mathbf{A}_{n+1}$, where $\mathbf{I}_{n+1}$ is the $(n+1) \times (n+1)$ identity matrix. Using partial fraction decomposition we find that

$$
\begin{aligned}
\frac{p'_{\mathbf{A}_{n+1}}(x)}{p_{\mathbf{A}_{n+1}}(x)} &= \sum_{0\le j\le n} \frac{p'_{\mathbf{A}_{n+1}}(\lambda_j)}{p'_{\mathbf{A}_{n+1}}(\lambda_j)} \frac{1}{x-\lambda_j} \\
&= \sum_{0\le j\le n} \frac{1}{x-\lambda_j} \\
&= \sum_{0\le j\le n} \frac{1}{x} \frac{x}{x-\lambda_j} \\
&= \sum_{0\le j\le n} \frac{1/x}{1-\lambda_j/x} \\
&= \sum_{0\le j\le n} \sum_{k\ge 0} \frac{1}{x} \left(\frac{\lambda_j}{x}\right)^k \\
&= \sum_{0\le j\le n} \sum_{k\ge 0} \frac{\lambda_j^k}{x^{k+1}} \\
&= \sum_{0\le j\le n} \sum_{k\ge 0} \lambda_j^k x^{-k-1} \\
&= \sum_{k\ge 0} x^{-k-1} \sum_{0\le j\le n} \lambda_j^k \\
&= \sum_{k\ge 0} x^{-k-1}\, \mathrm{tr}(\mathbf{A}_{n+1}^k) \\
&= \sum_{k\ge 0} x^{-k-1} \frac{F_{kn+k}}{F_k} && \text{by (30.7)} \\
&= \sum_{k\ge 0} x^{-k-1} \frac{\phi^{kn+k}-\hat{\phi}^{kn+k}}{\phi-\hat{\phi}} \frac{\phi-\hat{\phi}}{\phi^k-\hat{\phi}^k} && \text{by (30.1)} \\
&= \sum_{k\ge 0} x^{-k-1} \frac{\phi^{kn+k}-\hat{\phi}^{kn+k}}{\phi^k-\hat{\phi}^k} \\
&= \sum_{k\ge 0} x^{-k-1} \left(\hat{\phi}^k\right)^n \frac{1-\left(\phi^k/\hat{\phi}^k\right)^{n+1}}{1-\phi^k/\hat{\phi}^k} \\
&= \sum_{k\ge 0} x^{-k-1} \sum_{0\le j\le n} \left(\hat{\phi}^k\right)^n \left(\frac{\phi^k}{\hat{\phi}^k}\right)^j \\
&= \sum_{k\ge 0} x^{-k-1} \sum_{0\le j\le n} \phi^{jk}\hat{\phi}^{(n-j)k} \\
&= \sum_{0\le j\le n} \sum_{k\ge 0} \frac{1}{x} \left(\frac{\phi^j\hat{\phi}^{n-j}}{x}\right)^k \\
&= \sum_{0\le j\le n} \frac{1}{x} \frac{1}{1-\phi^j\hat{\phi}^{n-j}/x} \\
&= \sum_{0\le j\le n} \frac{1}{x-\phi^j\hat{\phi}^{n-j}}
\end{aligned}
$$

so that

$$\sum_{0\le j\le n} \frac{1}{x-\lambda_j} = \sum_{0\le j\le n} \frac{1}{x-\phi^j\hat{\phi}^{n-j}}$$
$$\iff \quad \lambda_j = \phi^j\hat{\phi}^{n-j}$$

and

$$p_{\mathbf{A}_{n+1}}(x) = \prod_{0\le j\le n}(x-\lambda_j) = \prod_{0\le j\le n}\left(x-\phi^j\hat{\phi}^{n-j}\right),$$

hence the result.

*Preliminary Result 30.9.* We may show that

$$(-1)^{k(k+1)/2} = (-1)^{\lceil k/2\rceil}. \tag{30.9}$$

In the case that $k = 2m$ even and $m$ even,

$$\begin{aligned}(-1)^{k(k+1)/2} &= (-1)^{2m(2m+1)/2}\\ &= (-1)^{m(2m+1)}\\ &= 1\\ &= (-1)^m\\ &= (-1)^{\lceil m\rceil}\\ &= (-1)^{\lceil 2m/2\rceil}\\ &= (-1)^{\lceil k/2\rceil};\end{aligned}$$

that $k = 2m$ even and $m$ odd,

$$\begin{aligned}(-1)^{k(k+1)/2} &= (-1)^{2m(2m+1)/2}\\ &= (-1)^{m(2m+1)}\\ &= -1\\ &= (-1)^m\\ &= (-1)^{\lceil m\rceil}\\ &= (-1)^{\lceil 2m/2\rceil}\\ &= (-1)^{\lceil k/2\rceil};\end{aligned}$$

that $k = 2m+1$ odd and $m$ even,

$$\begin{aligned}(-1)^{k(k+1)/2} &= (-1)^{(2m+1)(2m+2)/2}\\ &= (-1)^{(2m+1)(m+1)}\\ &= -1\\ &= (-1)^{m+1}\\ &= (-1)^{\lceil m+1/2\rceil}\\ &= (-1)^{\lceil (2m+1)/2\rceil}\\ &= (-1)^{\lceil k/2\rceil};\end{aligned}$$

and that $k = 2m + 1$ odd and $m$ odd,

$$\begin{aligned}(-1)^{k(k+1)/2} &= (-1)^{(2m+1)(2m+2)/2}\\ &= (-1)^{(2m+1)(m+1)}\\ &= 1\\ &= (-1)^{m+1}\\ &= (-1)^{\lceil m+1/2\rceil}\\ &= (-1)^{\lceil (2m+1)/2\rceil}\\ &= (-1)^{\lceil k/2\rceil};\end{aligned}$$

hence the result.

*Preliminary Result 30.10.* We may show that

$$\prod_{0\le j\le n}\left(x - \phi^j\hat{\phi}^{n-j}\right) = \sum_{0\le k\le n+1}(-1)^{\lceil k/2\rceil}\binom{n+1}{k}_{\mathcal{F}} x^{n+1-k}. \tag{30.10}$$

By the $q$-nomial theorem[2],

$$\prod_{0\le k\le n-1}\left(1 - q^k x\right) = \sum_{0\le k\le n}(-1)^k\binom{n}{k}_q q^{k(k-1)/2}x^k,$$

where

$$\binom{n}{k}_q = \prod_{1\le j\le k}\frac{1-q^{n-k+j}}{1-q^j},$$

[2]See exercise 1.2.6-58.

for $q = \hat\phi/\phi$,

$$\begin{aligned}
\binom{n}{k}_{\hat\phi/\phi} &= \prod_{1\le j\le k} \frac{1-(\hat\phi/\phi)^{n-k+j}}{1-(\hat\phi/\phi)^j} \\
&= \prod_{1\le j\le k} \frac{\left(\phi^{n-k+j}-\hat\phi^{n-k+j}\right)\phi^j}{\left(\phi^j-\hat\phi^j\right)\phi^{n-k+j}} \\
&= \prod_{1\le j\le k} \frac{\phi^j-\hat\phi^{n-k+j}\phi^{k-n}}{\phi^j-\hat\phi^j} \\
&= \prod_{1\le j\le k} \frac{\phi^j-\hat\phi^{n-k+j}\hat\phi^{n-k}}{\phi^j-\hat\phi^j} \\
&= \prod_{1\le j\le k} \frac{\phi^j-\hat\phi^{2n-2k+j}}{\phi^j-\hat\phi^j} \\
&= \prod_{1\le j\le k} \frac{\phi^j-\hat\phi^{n-k}\hat\phi^{n-k+j}}{\phi^j-\hat\phi^j} \\
&= \prod_{1\le j\le k} \frac{\phi^{k-n}\phi^{n-k+j}-\phi^{k-n}\hat\phi^{n-k+j}}{\phi^j-\hat\phi^j} \\
&= \prod_{1\le j\le k} \frac{\phi^{k-n}\left(\phi^{n-k+j}-\hat\phi^{n-k+j}\right)}{\phi^j-\hat\phi^j} \\
&= \left(\phi^{k-n}\right)^k \prod_{1\le j\le k} \frac{\phi^{n-k+j}-\hat\phi^{n-k+j}}{\phi^j-\hat\phi^j} \\
&= \phi^{k^2-nk} \prod_{1\le j\le k} \frac{\phi^{n-k+j}-\hat\phi^{n-k+j}}{\phi^j-\hat\phi^j} \\
&= \phi^{k^2-nk} \prod_{1\le j\le k} \frac{\phi^{n-k+j}-\hat\phi^{n-k+j}}{\phi-\hat\phi}\,\frac{\phi-\hat\phi}{\phi^j-\hat\phi^j} \\
&= \phi^{k^2-nk} \prod_{1\le j\le k} \frac{F_{n-k+j}}{F_j} && \text{by (30.1)} \\
&= \phi^{k^2-nk}\binom{n}{k}_{\mathcal{F}}.
\end{aligned}$$

And so,

$$\begin{aligned}
&\prod_{0\le k\le n-1}\left(1-(\hat\phi/\phi)^k x\right) \\
&= \prod_{0\le k\le n-1}\left(1-\phi^{-k}\hat\phi^k x\right) \\
&= \sum_{0\le k\le n} (-1)^k \binom{n}{k}_{\hat\phi/\phi} (\hat\phi/\phi)^{k(k-1)/2} x^k \\
&= \sum_{0\le k\le n} (-1)^k \phi^{k^2-nk}\binom{n}{k}_{\mathcal{F}} (\hat\phi/\phi)^{k(k-1)/2} x^k \\
&= \sum_{0\le k\le n} (-1)^k \phi^{k(k+1)/2-nk}\hat\phi^{k(k-1)/2}\binom{n}{k}_{\mathcal{F}} x^k.
\end{aligned}$$

Substituting $\phi^{n-1}x$ for $x$ yields

$$\begin{aligned}
&\prod_{0\le k\le n-1}\Big(1-\phi^{-k}\hat\phi^{k}\phi^{n-1}x\Big)\\
&=\prod_{0\le k\le n-1}\Big(1-\phi^{n-k-1}\hat\phi^{k}x\Big)\\
&=\sum_{0\le k\le n}(-1)^k\phi^{k(k+1)/2-nk}\hat\phi^{k(k-1)/2}\binom{n}{k}_{\mathcal F}(\phi^{n-1}x)^k\\
&=\sum_{0\le k\le n}(-1)^k\phi^{k(k+1)/2-nk}\hat\phi^{k(k-1)/2}\binom{n}{k}_{\mathcal F}\phi^{kn-k}x^k\\
&=\sum_{0\le k\le n}(-1)^k\phi^{k(k+1)/2-k}\hat\phi^{k(k-1)/2}\binom{n}{k}_{\mathcal F}x^k\\
&=\sum_{0\le k\le n}(-1)^k\phi^{k(k-1)/2}\hat\phi^{k(k-1)/2}\binom{n}{k}_{\mathcal F}x^k\\
&=\sum_{0\le k\le n}(-1)^k(\phi\hat\phi)^{k(k-1)/2}\binom{n}{k}_{\mathcal F}x^k\\
&=\sum_{0\le k\le n}(-1)^k(-1)^{k(k-1)/2}\binom{n}{k}_{\mathcal F}x^k\\
&=\sum_{0\le k\le n}(-1)^{k(k+1)/2}\binom{n}{k}_{\mathcal F}x^k.
\end{aligned}$$

Substituting $1/x$ for $x$ yields

$$\begin{aligned}
&\prod_{0\le k\le n-1}\Big(1-\phi^{n-k-1}\hat\phi^{k}/x\Big)\\
&=\prod_{0\le k\le n-1}\frac{x-\phi^{n-k-1}\hat\phi^{k}}{x}\\
&=\frac{1}{x^n}\prod_{0\le k\le n-1}\Big(x-\phi^{n-k-1}\hat\phi^{k}\Big)\\
&=\sum_{0\le k\le n}(-1)^{k(k+1)/2}\binom{n}{k}_{\mathcal F}(1/x)^k\\
&=\sum_{0\le k\le n}(-1)^{k(k+1)/2}\binom{n}{k}_{\mathcal F}\frac{1}{x^k}\\
&=\sum_{0\le k\le n}(-1)^{\lceil k/2\rceil}\binom{n}{k}_{\mathcal F}\frac{1}{x^k} \qquad \text{by (30.9)}
\end{aligned}$$

if and only if

$$\prod_{0\le k\le n-1}\Big(x-\phi^{n-k-1}\hat\phi^{k}\Big)=\sum_{0\le k\le n}(-1)^{\lceil k/2\rceil}\binom{n}{k}_{\mathcal F}x^{n-k},$$

or equivalently,

$$\prod_{0\le j\le n}\Big(x-\phi^{j}\hat\phi^{n-j}\Big)=\sum_{0\le k\le n+1}(-1)^{\lceil k/2\rceil}\binom{n+1}{k}_{\mathcal F}x^{n+1-k},$$

and hence the result.

*Preliminary Result 30.11.* We may show that

$$\left[\mathbf{A}_{n+1}^{k}\right]_{n,j} = \binom{n}{j} F_{k+1}^{j} F_{k}^{n-j}. \tag{30.11}$$

In the case that $k = 0$,

$$\begin{aligned}\left[\mathbf{A}_{n+1}^{0}\right]_{n,j} &= \delta_{nj} \\ &= \binom{n}{j} F_{0+1}^{j} F_{0}^{n-j}.\end{aligned}$$

Then, assuming

$$\left[\mathbf{A}_{n+1}^{k}\right]_{n,j} = \binom{n}{j} F_{k+1}^{j} F_{k}^{n-j},$$

we must show that

$$\left[\mathbf{A}_{n+1}^{k+1}\right]_{n,j} = \binom{n}{j} F_{k+2}^{j} F_{k+1}^{n-j}.$$

But

$$\begin{aligned}
&\left[\mathbf{A}_{n+1}^{k+1}\right]_{n,j} \\
&\quad = \left[\mathbf{A}_{n+1}^{k} \cdot \mathbf{A}_{n+1}\right]_{n,j} \\
&\quad = \sum_{0 \le m \le n} \left[\mathbf{A}_{n+1}^{k}\right]_{n,m} \left[\mathbf{A}_{n+1}\right]_{m,j} \\
&\quad = \sum_{0 \le m \le n} \binom{n}{m} F_{k+1}^{m} F_{k}^{n-m} \left[\mathbf{A}_{n+1}\right]_{m,j} \\
&\quad = \sum_{0 \le m \le n} \binom{n}{m} F_{k+1}^{m} F_{k}^{n-m} \binom{m}{n-j} \\
&\quad = \sum_{0 \le m \le n} \binom{n}{n-j} \binom{n-(n-j)}{m-(n-j)} F_{k+1}^{m} F_{k}^{n-m} && \text{by Eq. (1.2.6-2)} \\
&\quad = \sum_{0 \le m \le n} \binom{n}{n-j} \binom{j}{j+m-n} F_{k+1}^{m} F_{k}^{n-m} \\
&\quad = \sum_{0 \le m \le n} \binom{n}{n-j} F_{k+1}^{n-j} \binom{j}{j+m-n} F_{k+1}^{j+m-n} F_{k}^{n-m} \\
&\quad = \binom{n}{j} F_{k+1}^{n-j} \sum_{0 \le m \le j} \binom{j}{m} F_{k+1}^{m} F_{k}^{j-m} \\
&\quad = \binom{n}{j} F_{k+1}^{n-j} \left(F_{k+1} + F_{k}\right)^{j} \\
&\quad = \binom{n}{j} F_{k+2}^{j} F_{k+1}^{n-j}
\end{aligned}$$

and hence the result.

*Conclusion.* We will now show that

$$\sum_{0 \le k \le m} \binom{m}{k}_{\mathcal{F}} (-1)^{\lceil k/2 \rceil} F_{n'-k}^{m-1} = 0.$$

From (30.8) and (30.10), the characteristic polynomial satisifes

$$p_{\mathbf{A}_{n+1}}(x) = \prod_{0\le j\le n}\left(x - \phi^j\hat{\phi}^{n-j}\right) = \sum_{0\le k\le n+1}(-1)^{\lceil k/2\rceil}\binom{n+1}{k}_{\mathcal{F}} x^{n+1-k}.$$

But, by the *Cayley-Hamilton* theorem, every matrix satisifes its characteristic polynomial. And so,

$$\sum_{0\le k\le n+1}(-1)^{\lceil k/2\rceil}\binom{n+1}{k}_{\mathcal{F}}\mathbf{A}_{n+1}^{n'-1-k} = \mathcal{O}$$

for $n'-1\ge n+1$, where $\mathcal{O}$ denotes the $(n+1)\times(n+1)$ zero matrix. By (30.11), for $n=j$ and $k=n'-1-k$,

$$\left[\mathbf{A}_{n+1}^{n'-1-k}\right]_{n,n} = \binom{n}{n}F_{n'-1-k+1}^{n}F_{n'-1-k}^{n-n} = F_{n'-k}^{n}.$$

And so,

$$\sum_{0\le k\le n+1}(-1)^{\lceil k/2\rceil}\binom{n+1}{k}_{\mathcal{F}}F_{n'-k}^{n} = 0,$$

or equivalently,

$$\sum_{0\le k\le m}\binom{m}{k}_{\mathcal{F}}(-1)^{\lceil k/2\rceil}F_{n'-k}^{m-1} = 0.$$

This concludes the proof. □

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[D. Jarden, *Recurring Sequences*, 2nd ed. (Jerusalem, 1966), 30–33; J. Riordan, *Duke Math. J.* **29** (1962), 5–12]

**31.** [*M20*] Show that $F_{2n}\phi \bmod 1 = 1-\phi^{-2n}$ and $F_{2n+1}\phi \bmod 1 = \phi^{-2n-1}$.

We may show both identities.

**Proposition.** $F_{2n}\phi \bmod 1 = 1-\phi^{-2n}$.

*Proof.* Let $n$ be an arbitrary integer. We must show that

$$F_{2n}\phi \bmod 1 = 1-\phi^{-2n},$$

or equivalently that

$$1 \mid F_{2n}\phi - (1-\phi^{-2n}).$$

But clearly, $1\mid F_{2n+1}-1$, and

$$\begin{aligned}
F_{2n+1}-1 &= F_{2n}\phi - F_{2n}\phi + F_{2n+1} - 1\\
&= F_{2n}\phi + (-1)^{2n+1}F_{2n}\phi + (-1)^{2(n+1)}F_{2n+1} - 1\\
&= F_{2n}\phi + (-1)^{2n+1}F_{2n}\phi + (-1)^{2n+2}F_{2n+1} - 1\\
&= F_{2n}\phi + F_{-2n}\phi + F_{-(2n+1)} - 1 && \text{by exercise 8}\\
&= F_{2n}\phi - 1 + F_{-2n}\phi + F_{-2n-1}\\
&= F_{2n}\phi - 1 + \phi^{-2n} && \text{by exercise 11}\\
&= F_{2n}\phi - (1-\phi^{-2n}).
\end{aligned}$$

That is,

$$1 \mid F_{2n}\phi - (1-\phi^{-2n})$$

as we needed to show. □

**Proposition.** $F_{2n+1}\phi \bmod 1 = \phi^{-2n-1}$.

*Proof.* Let $n$ be an arbitrary integer. We must show that

$$F_{2n+1}\phi \bmod 1 = \phi^{-2n-1},$$

or equivalently that

$$1 \mid F_{2n+1}\phi - \phi^{-2n-1}.$$

But clearly, $1 \mid F_{2n+2}$, and

$$\begin{aligned}
F_{2n+2} &= F_{2n+1}\phi - F_{2n+1}\phi + F_{2n+2} \\
&= F_{2n+1}\phi - (-1)^{2(n+1)}F_{2n+1}\phi - (-1)^{2(n+1)+1}F_{2n+2} \\
&= F_{2n+1}\phi - (-1)^{2n+2}F_{2n+1}\phi - (-1)^{2n+3}F_{2n+2} \\
&= F_{2n+1}\phi - F_{-(2n+1)}\phi - F_{-(2n+2)} && \text{by exercise 8} \\
&= F_{2n+1}\phi - (F_{-2n-1}\phi + F_{-2n-2}) \\
&= F_{2n+1}\phi - \phi^{-2n-1}. && \text{by exercise 11}
\end{aligned}$$

That is,

$$1 \mid F_{2n+1}\phi - \phi^{-2n-1}$$

as we needed to show. □

**32.** [*M24*] The remainder of one Fibonacci number divided by another is ± a Fibonacci number: Show that, modulo $F_n$,

$$F_{mn+r} \equiv \begin{cases} F_r, & \text{if } m \bmod 4 = 0; \\ (-1)^{r+1}F_{n-r}, & \text{if } m \bmod 4 = 1; \\ (-1)^n F_r, & \text{if } m \bmod 4 = 2; \\ (-1)^{r+1+n}F_{n-r}, & \text{if } m \bmod 4 = 3. \end{cases}$$

**Proposition.** $F_{mn+r} \equiv \begin{cases} F_r & \textit{if } m \bmod 4 = 0 \\ (-1)^{r+1}F_{n-r} & \textit{if } m \bmod 4 = 1 \\ (-1)^n F_r & \textit{if } m \bmod 4 = 2 \\ (-1)^{r+1+n}F_{n-r} & \textit{if } m \bmod 4 = 3 \end{cases} \pmod{F_n}$.

*Proof.* Let $m, n, r$ be arbitrary integers, such that $m = n = r = 0$ or $0 \le r < m \le n$. We must show that

$$F_{mn+r} \equiv \begin{cases} F_r & \text{if } m \bmod 4 = 0 \\ (-1)^{r+1}F_{n-r} & \text{if } m \bmod 4 = 1 \\ (-1)^n F_r & \text{if } m \bmod 4 = 2 \\ (-1)^{r+1+n}F_{n-r} & \text{if } m \bmod 4 = 3 \end{cases} \pmod{F_n}.$$

As preliminaries, note that since

$$F_n \equiv 0 \pmod{F_n},$$

by repeated applications of Law 1.2.4-A,

$$aF_n \equiv 0 \pmod{F_n},$$

for any integer $a$, which will be hereafter indicated as *by Law 1.2.4-A**. Also

$$\begin{aligned}
F_n &\equiv 0 \\
&\equiv F_n \\
&\equiv F_{n+1} - F_{n-1} \pmod{F_n}
\end{aligned}$$

if and only if

$$F_{n+1} \equiv F_{n-1} \pmod{F_n}. \tag{32.1}$$

In the case that $m \bmod 4 = 0$, we have that

$$\begin{aligned}
F_r &\equiv F_r \\
&\equiv F_{n+1}^0 F_r \\
&\equiv F_{n+1}^{m \bmod 4} F_r \pmod{F_n}.
\end{aligned}$$

In the case that $m \bmod 4 = 1$, we have that

$$\begin{aligned}
(-1)^{r+1} F_{n-r} &\equiv (-1)^{r+1} F_{n-r} F_1 \\
&\equiv (-1)^{r+1} F_{n-r} (-1)^2 F_1 \\
&\equiv (-1)^{r+1} F_{n-r} (-1)^{1+1} F_1 \\
&\equiv (-1)^{r+1} F_{n-r} F_{-1} && \text{by exercise 8} \\
&\equiv (-1)^{r+1} F_{n-(r+1)-(-1)} F_{-1} \\
&\equiv F_{(r+1)+(-1)} F_{n-(-1)} - F_{r+1} F_n && \text{by exercise 17} \\
&\equiv F_r F_{n+1} - F_{r+1} F_n \\
&\equiv F_r F_{n+1} && \text{by Law 1.2.4-A*} \\
&\equiv F_{n+1}^1 F_r \\
&\equiv F_{n+1}^{m \bmod 4} F_r \pmod{F_n}.
\end{aligned}$$

In the case that $m \bmod 4 = 2$, we have that

$$\begin{aligned}
(-1)^n F_r &\equiv (-1)^n F_1 F_r \\
&\equiv (-1)^n (-1)^2 F_1 F_r \\
&\equiv (-1)^n (-1)^{1+1} F_1 F_r \\
&\equiv (-1)^n F_{-1} F_1 F_r && \text{by exercise 8} \\
&\equiv (-1)^n F_{n-n-1} F_1 F_r \\
&\equiv (F_{n+1} F_{n-1} - F_n F_n) F_r && \text{by exercise 17} \\
&\equiv F_{n+1} F_{n-1} F_r - F_n^2 F_r \\
&\equiv F_{n+1} F_{n-1} F_r && \text{by Law 1.2.4-A*} \\
&\equiv F_{n+1} F_{n+1} F_r && \text{by (32.1)} \\
&\equiv F_{n+1}^2 F_r \\
&\equiv F_{n+1}^{m \bmod 4} F_r \pmod{F_n}.
\end{aligned}$$

In the case that $m \bmod 4 = 3$, we have that

$$\begin{aligned}
(-1)^{r+1+n}F_{n-r} &\equiv (-1)^n(-1)^{r+1}F_{n-r} \\
&\equiv (-1)^n(-1)^{r+1}F_{n-r}F_1 \\
&\equiv (-1)^n(-1)^{r+1}F_{n-r}(-1)^2F_1 \\
&\equiv (-1)^n(-1)^{r+1}F_{n-r}(-1)^{1+1}F_1 \\
&\equiv (-1)^n(-1)^{r+1}F_{n-r}F_{-1} && \text{by exercise 8} \\
&\equiv (-1)^n(-1)^{r+1}F_{n-(r+1)-(-1)}F_{-1} \\
&\equiv (-1)^nF_{(r+1)+(-1)}F_{n-(-1)} - F_{r+1}F_n && \text{by exercise 17} \\
&\equiv (-1)^nF_rF_{n+1} - F_{r+1}F_n \\
&\equiv (-1)^nF_rF_{n+1} && \text{by Law 1.2.4-A*} \\
&\equiv (-1)^nF_1F_rF_{n+1} \\
&\equiv (-1)^n(-1)^2F_1F_rF_{n+1} \\
&\equiv (-1)^n(-1)^{1+1}F_1F_rF_{n+1} \\
&\equiv (-1)^nF_{-1}F_1F_rF_{n+1} && \text{by exercise 8} \\
&\equiv (-1)^nF_{n-n-1}F_1F_rF_{n+1} \\
&\equiv (F_{n+1}F_{n-1} - F_nF_n)F_rF_{n+1} && \text{by exercise 17} \\
&\equiv F_{n+1}^2F_{n-1}F_r - F_n^2F_rF_{n+1} \\
&\equiv F_{n+1}^2F_{n-1}F_r && \text{by Law 1.2.4-A*} \\
&\equiv F_{n+1}^2F_{n+1}F_r && \text{by (32.1)} \\
&\equiv F_{n+1}^3F_r \\
&\equiv F_{n+1}^{m \bmod 4}F_r \pmod{F_n}.
\end{aligned}$$

That is, we must show that

$$F_{mn+r} \equiv F_{n+1}^{m \bmod 4}F_r \pmod{F_n}.$$

If $m = 0$, then $n = m = r = 0$, $m \bmod 4 = 0$, and

$$\begin{aligned}
F_{mn+r} &\equiv F_{0\cdot 0+0} \\
&\equiv F_0 \\
&\equiv F_1^0F_0 \\
&\equiv F_{0+1}^0F_0 \\
&\equiv F_{n+1}^{m \bmod 4}F_r \pmod{F_n}.
\end{aligned}$$

If $m = 1$, then $m \bmod 4 = 1$, and

$$\begin{aligned}
F_{mn+r} &\equiv F_{1\cdot n+r} \\
&\equiv F_{n+r} \\
&\equiv F_rF_{n+1} + F_{r-1}F_n && \text{by Eq. (4)} \\
&\equiv F_rF_{n+1} && \text{by Law 1.2.4-A*} \\
&\equiv F_{n+1}^1F_r \\
&\equiv F_{n+1}^{m \bmod 4}F_r \pmod{F_n}.
\end{aligned}$$

If $m = 2$, then $m \bmod 4 = 2$, and

$$\begin{aligned}
F_{mn+r} &\equiv F_{2\cdot n+r} \\
&\equiv F_{n+n+r} \\
&\equiv F_{n+r}F_{n+1} + F_{n+r-1}F_n && \text{by Eq. (4)} \\
&\equiv F_{n+r}F_{n+1} && \text{by Law 1.2.4-A*} \\
&\equiv (F_rF_{n+1} + F_{r-1}F_n)F_{n+1} && \text{by Eq. (4)} \\
&\equiv F_rF_{n+1}^2 + F_{r-1}F_{n+1}F_n \\
&\equiv F_rF_{n+1}^2 && \text{by Law 1.2.4-A*} \\
&\equiv F_{n+1}^2F_r \\
&\equiv F_{n+1}^{m \bmod 4}F_r \pmod{F_n}.
\end{aligned}$$

If $m = 3$, then $m \bmod 4 = 3$, and

$$\begin{aligned}
F_{mn+r} &\equiv F_{3\cdot n+r} \\
&\equiv F_{n+2n+r} \\
&\equiv F_{2n+r}F_{n+1} + F_{2n+r-1}F_n && \text{by Eq. (4)} \\
&\equiv F_{2n+r}F_{n+1} && \text{by Law 1.2.4-A*} \\
&\equiv F_{n+n+r}F_{n+1} \\
&\equiv (F_{n+r}F_{n+1} + F_{n+r-1}F_n)F_{n+1} && \text{by Eq. (4)} \\
&\equiv F_{n+r}F_{n+1}^2 + F_{n+r-1}F_{n+1}F_n \\
&\equiv F_{n+r}F_{n+1}^2 && \text{by Law 1.2.4-A*} \\
&\equiv (F_rF_{n+1} + F_{r-1}F_n)F_{n+1}^2 && \text{by Eq. (4)} \\
&\equiv F_rF_{n+1}^3 + F_{r-1}F_{n+1}^2F_n \\
&\equiv F_rF_{n+1}^3 && \text{by Law 1.2.4-A*} \\
&\equiv F_{n+1}^3F_r \\
&\equiv F_{n+1}^{m \bmod 4}F_r \pmod{F_n}.
\end{aligned}$$

Then, assuming

$$F_{mn+r} \equiv F_{n+1}^{m \bmod 4}F_r \pmod{F_n},$$

we must show that

$$F_{(m+1)n+r} \equiv F_{n+1}^{(m+1) \bmod 4}F_r \pmod{F_n}.$$

But

$$\begin{aligned}
F_{(m+1)n+r} &\equiv F_{mn+n+r} \\
&\equiv F_{n+mn+r} \\
&\equiv F_{mn+r}F_{n+1} + F_{mn+r-1}F_n && \text{by Eq. (4)} \\
&\equiv F_{mn+r}F_{n+1} && \text{by Law 1.2.4-A*} \\
&\equiv F_{n+1}^{m \bmod 4}F_rF_{n+1} \\
&\equiv F_{n+1}^{m \bmod 4+1}F_r \pmod{F_n}.
\end{aligned}$$

Here, we divide the proof into cases depending on $m \bmod 4$. In the case that $m \bmod 4 =$

0, $(m+1) \bmod 4 = 1$ and

$$\begin{aligned} F_{n+1}^{m \bmod 4+1} F_r &\equiv F_{n+1}^{0+1} F_r \\ &\equiv F_{n+1}^{1} F_r \\ &\equiv F_{n+1}^{(m+1) \bmod 4} F_r \pmod{F_n}. \end{aligned}$$

In the case that $m \bmod 4 = 1$, $(m+1) \bmod 4 = 2$ and

$$\begin{aligned} F_{n+1}^{m \bmod 4+1} F_r &\equiv F_{n+1}^{1+1} F_r \\ &\equiv F_{n+1}^{2} F_r \\ &\equiv F_{n+1}^{(m+1) \bmod 4} F_r \pmod{F_n}. \end{aligned}$$

In the case that $m \bmod 4 = 2$, $(m+1) \bmod 4 = 3$ and

$$\begin{aligned} F_{n+1}^{m \bmod 4+1} F_r &\equiv F_{n+1}^{2+1} F_r \\ &\equiv F_{n+1}^{3} F_r \\ &\equiv F_{n+1}^{(m+1) \bmod 4} F_r \pmod{F_n}. \end{aligned}$$

In the case that $m \bmod 4 = 3$, $(m+1) \bmod 4 = 0$ and

$$\begin{aligned} F_{n+1}^{m \bmod 4+1} F_r &\equiv F_{n+1}^{3+1} F_r \\ &\equiv F_{n+1}^{4} F_r \\ &\equiv \left(F_{n+1} F_{n+1}\right)^2 F_r \\ &\equiv \left(F_{n+1} F_{n-1}\right)^2 F_r && \text{by (32.1)} \\ &\equiv \left(F_{n+1} F_{n-1}\right)^2 F_r + F_n F_r \left(-2F_n F_{n+1} F_{n-1} + F_n^3\right) && \text{by Law 1.2.4-A*} \\ &\equiv \left(\left(F_{n+1} F_{n-1}\right)^2 + F_n \left(-2F_n F_{n+1} F_{n-1} + F_n^3\right)\right) F_r \\ &\equiv \left(\left(F_{n+1} F_{n-1}\right)^2 - 2F_n^2 F_{n+1} F_{n-1} + \left(F_n^2\right)^2\right) F_r \\ &\equiv \left(F_{n+1} F_{n-1} - F_n^2\right)^2 F_r \\ &\equiv \left(F_{n+1} F_{n-1} - F_n F_n\right)^2 F_r \\ &\equiv \left((-1)^n F_{n-n-1} F_1\right)^2 F_r && \text{by exercise 17} \\ &\equiv \left((-1)^n F_{-1} F_1\right)^2 F_r \\ &\equiv \left((-1)^n F_{-1}\right)^2 F_r \\ &\equiv \left((-1)^n (-1)^{1+1} F_1\right)^2 F_r && \text{by exercise 8} \\ &\equiv \left((-1)^n (-1)^2 F_1\right)^2 F_r \\ &\equiv \left((-1)^n (-1)^2\right)^2 F_r \\ &\equiv \left((-1)^n\right)^2 F_r \\ &\equiv (-1)^{2n} F_r \\ &\equiv \left((-1)^2\right)^n F_r \\ &\equiv 1^n F_r \\ &\equiv 1 \cdot F_r \\ &\equiv F_{n+1}^0 F_r \\ &\equiv F_{n+1}^{(m+1) \bmod 4} F_r \pmod{F_n}. \end{aligned}$$

And so,

$$\begin{aligned}F_{(m+1)n+r} &\equiv F_{n+1}^{m \bmod 4+1} F_r \\ &\equiv F_{n+1}^{(m+1) \bmod 4} F_r \pmod{F_n}\end{aligned}$$

as we needed to show. □

**33.** [*HM24*] Given that $z = \pi/2 + i \ln \phi$, show that $\sin nz / \sin z = i^{1-n} F_n$.

**Proposition.** $\sin(nz)/\sin(z) = i^{1-n} F_n$ *if* $z = \pi/2 + i \ln \phi$.

*Proof.* Let $n$ be an arbitrary nonnegative integer, and $z = \pi/2 + i \ln \phi$. We must show that

$$\sin(nz)/\sin(z) = i^{1-n} F_n.$$

As preliminaries, note that

$$\begin{aligned}\cos(z) &= \frac{1}{2}(e^{iz} + e^{-iz}) \\ &= \frac{1}{2}(e^{i(\pi/2+i\ln\phi)} + e^{-i(\pi/2+i\ln\phi)}) \\ &= \frac{1}{2}(e^{i\pi/2+i^2\ln\phi} + e^{-(i\pi/2+i^2\ln\phi)}) \\ &= \frac{1}{2}(e^{i\pi/2-\ln\phi} + e^{-i\pi/2+\ln\phi}) \\ &= \frac{1}{2}(e^{i\pi/2}e^{-\ln\phi} + e^{-i\pi/2}e^{\ln\phi}) \\ &= \frac{1}{2}(\sqrt{e^{i\pi}}(e^{\ln\phi})^{-1} + (\sqrt{e^{i\pi}})^{-1}e^{\ln\phi}) \\ &= \frac{1}{2}(\sqrt{-1}(\phi)^{-1} + (\sqrt{-1})^{-1}\phi) \\ &= \frac{1}{2}(i(\phi)^{-1} + (i)^{-1}\phi) \\ &= \frac{1}{2}(i/\phi + \phi/i) \\ &= \frac{1}{2}(2i/(1+\sqrt{5}) + (1+\sqrt{5})/2i) \\ &= \frac{1}{2}((2i)^2/2i(1+\sqrt{5}) + (1+\sqrt{5})^2/2i(1+\sqrt{5})) \\ &= \frac{1}{2}(-4/(2i+2i\sqrt{5}) + (1+2\sqrt{5}+5)/(2i+2i\sqrt{5})) \\ &= \frac{1}{2}(-4+(1+2\sqrt{5}+5))/(2i+2i\sqrt{5}) \\ &= \frac{1}{2}(-4+1+2\sqrt{5}+5)/(2i+2i\sqrt{5}) \\ &= \frac{1}{2}(2+2\sqrt{5})/(2i+2i\sqrt{5}) \\ &= (1+\sqrt{5})/2i(1+\sqrt{5}) \\ &= 1/2i \\ &= -i/2\end{aligned}$$

and

$$\begin{aligned} 2\sin(nz)\cos(z) &= \sin(nz+z)+\sin(nz-z) \\ &= \sin((n+1)z)+\sin((n-1)z) \end{aligned}$$

so that

$$\begin{aligned} & 2\sin(nz)\cos(z) = \sin((n+1)z)+\sin((n-1)z) \\ \iff\quad & -2i\sin(nz)/2 = \sin((n+1)z)+\sin((n-1)z) \\ \iff\quad & -i\sin(nz) = \sin((n+1)z)+\sin((n-1)z) \\ \iff\quad & \sin(nz) = -(\sin((n+1)z)+\sin((n-1)z))/i \\ \iff\quad & \sin(nz) = i(\sin((n+1)z)+\sin((n-1)z)) \\ \iff\quad & \sin(nz)/\sin(z) = i(\sin((n+1)z)+\sin((n-1)z))/sin(z). \end{aligned}$$

If $n = 0$,

$$\begin{aligned} \sin(nz)/\sin(z) &= i(\sin((n+1)z)+\sin((n-1)z))/sin(z) \\ &= i(\sin(z)+\sin(-z))/sin(z) \\ &= i(\sin(z)-\sin(z))/sin(z) \\ &= i\cdot 0 \\ &= i^1 F_0 \\ &= i^{1-n}F_n; \end{aligned}$$

and if $n = 1$,

$$\begin{aligned} \sin(nz)/\sin(z) &= i(\sin((n+1)z)+\sin((n-1)z))/sin(z) \\ &= i(\sin(2z)+\sin(0))/sin(z) \\ &= i\sin(2z)/sin(z) \\ &= 2i\cos(z)\sin(z)/\sin(z) \\ &= 2i\cos(z) \\ &= -2i^2/2 \\ &= -i^2 \\ &= -(-1) \\ &= 1 \\ &= i^0 \\ &= i^{1-1}F_1 \\ &= i^{1-n}F_n. \end{aligned}$$

Then, assuming that

$$\sin(nz)/\sin(z) = i^{1-n}F_n,$$

we must show that

$$\sin((n+1)z)/\sin(z) = i^{1-(n+1)}F_{n+1}.$$

But

$$\begin{aligned}
&\sin((n+1)z)/\sin(z)\\
&\quad= (2\cos(z)\sin(nz) - \sin((n-1)z))/\sin(z)\\
&\quad= (-2i\sin(nz)/2 - \sin((n-1)z))/\sin(z)\\
&\quad= (-i\sin(nz) - \sin((n-1)z))/\sin(z)\\
&\quad= i^{-1}\sin(nz)/\sin(z) - \sin((n-1)z)/\sin(z)\\
&\quad= i^{-1}i^{1-n}F_n - i^{1-(n-1)}F_{n-1}\\
&\quad= i^{1-(n+1)}F_n - i^{1-n+1}F_{n-1}\\
&\quad= i^{1-(n+1)}F_n + i^{1-n-1}F_{n-1}\\
&\quad= i^{1-(n+1)}F_n + i^{1-(n+1)}F_{n-1}\\
&\quad= i^{1-(n+1)}(F_n + F_{n-1})\\
&\quad= i^{1-(n+1)}F_{n+1}
\end{aligned}$$

as we needed to show. □

► **34.** [*M24*] (*The Fibonacci number system.*) Let the notation $k \gg m$ mean that $k \geq m+2$. Show that every positive integer $n$ has a *unique* representation $n = F_{k_1}+F_{k_2}+\cdots+F_{k_r}$, where $k_1 \gg k_2 \gg \cdots \gg k_r \gg 0$.

First, we prove a corollary.

**Proposition.** $\sum_{1\leq j\leq r} F_{k_j} < F_{k_r+1}$, *where* $k_j > k_{j+1}+1$ *for* $1 \leq j < r$ *and* $k_r > 1$.

*Proof.* Let $r$ be an arbitrary positive integer. We must show that

$$\sum_{1\leq j\leq r} F_{k_j} < F_{k_1+1} \tag{34.1}$$

where $k_j > k_{j+1}+1$ for $1 \leq j < r$ and $k_r > 1$. If $r = 1$,

$$\sum_{1\leq j\leq 1} F_{k_j} = F_{k_1} = F_2 = 1 < 2 = F_3 = F_{2+1} = F_{k_1+1}$$

where $k_1 = 2 > 1$. Then, assuming

$$\sum_{1\leq j\leq r} F_{k_j} < F_{k_1+1}$$

where $k_j > k_{j+1}+1$ for $1 \leq j < r$ and $k_r > 1$, we must show that

$$\sum_{1\leq j\leq r'} F_{k'_j} < F_{k'_1+1}$$

where $r' = r+1$, $k'_j > k'_{j+1}+1$ for $1 \leq j < r'$ and $k_{r'} > 1$. But since

$$\begin{aligned}
k'_1 > k'_2 + 1 \quad&\iff\quad k'_1 \geq k'_2 + 2\\
&\iff\quad k'_1 - 1 \geq k'_2 + 1\\
&\iff\quad F_{k'_2+1} \leq F_{k'_1-1}
\end{aligned}$$

then

$$\begin{aligned}
\sum_{1\leq j\leq r'} F_{k'_j} &= F_{k'_1} + \sum_{2\leq j\leq r'} F_{k'_j}\\
&< F_{k'_1} + F_{k'_2+1}\\
&\leq F_{k'_1} + F_{k'_1-1}\\
&= F_{k'_1+1}
\end{aligned}$$

as we needed to show. □

Then, we proceed with the requested proof.

**Proposition.** *Every positive integer $n$ has a unique representation $n = \sum_{1 \le j \le r} F_{k_j}$, where $k_j > k_{j+1} + 1$ for $1 \le j < r$ and $k_r > 1$.*

*Proof.* Let $n$ be an arbitrary positive integer. We must show that for $n$ there *exists* a representation

$$n = \sum_{1 \le j \le r} F_{k_j}$$

where $k_j > k_{j+1} + 1$ for $1 \le j < r$ and $k_r > 1$; and that this representation is *unique*.

*Existence.* If $n = 1$,

$$1 = \sum_{1 \le j \le 1} F_{k_j} = F_{k_1} = F_2$$

where $k_1 = 2 > 1$; if $n = 2$,

$$2 = \sum_{1 \le j \le 1} F_{k_j} = F_{k_1} = F_3$$

where $k_1 = 3 > 1$; if $n = 3$,

$$3 = \sum_{1 \le j \le 1} F_{k_j} = F_{k_1} = F_4$$

where $k_1 = 4 > 1$; and if $n = 4$,

$$4 = \sum_{1 \le j \le 2} F_{k_j} = F_{k_1} + F_{k_2} = F_4 + F_2 = 3 + 1$$

where $k_1 = 4 > 2+1 = k_2+1$, $k_2 = 2 > 1$. Then, assuming there exists a representation

$$n = \sum_{1 \le j \le r} F_{k_j},$$

we must show that there exists a representation

$$n + 1 = \sum_{1 \le j \le r'} F_{k'_j}.$$

In the case that $n + 1$ is a Fibonacci number $F_{k'}$, we have that

$$n + 1 = \sum_{1 \le j \le 1} F_{k'_j} = F_{k'_1} = F_{k'}$$

where $k'_1 = k' > 1$. Otherwise, in the case that $n + 1$ is not a Fibonacci number, it must lie between two Fibonacci numbers. That is, there must exist a $j'$ such that

$$F_{j'} < n + 1 < F_{j'+1}.$$

Let $m = n + 1 - F_{j'}$. Since $m \le n$, by the inductive hypothesis, there exists a representation

$$m = \sum_{1 \le j \le r} F_{k_j}.$$

But

$$\begin{aligned} n + 1 < F_{j'+1} \quad &\iff \quad m + F_{j'} < F_{j'+1} \\ &\iff \quad m < F_{j'+1} - F_{j'} \\ &\iff \quad m < F_{j'-1}. \end{aligned}$$

That is, $m$ does not contain $F_{j'-1}$, and so

$$n+1 = \sum_{1\le j\le r'} F_{k'_j} = F_{j'} + m = F_{j'} + \sum_{1\le j\le r} F_{k_j}$$

where $r' = r+1$, $k'_j = k_{j-1}$ if $2 \le j \le r'$, $k'_1 = j'$ otherwise, and $k'_1 = j' > k_1 + 1$, and hence existence.

*Uniqueness.* Let $n$ have the two representations

$$n = \sum_{1\le j\le r} F_{k_j}$$

and

$$n = \sum_{1\le j\le r'} F_{k'_j};$$

and let the Fibonacci numbers of each be represented by the sets $S = \{F_{k_j}\}$ and $S' = \{F_{k'_j}\}$. The previous result has shown that they each contain non-consecutive Fibonacci numbers, and have the same cardinality: $r = r'$. Then let $T = S - S'$ and $T' = S' - S$. Since $\sum_{s\in S} s = \sum_{s'\in S'} s'$, we also have that $\sum_{t\in T} t = \sum_{t'\in T'} t'$. We will assume that $S \ne S'$, so that neither $T$ nor $T'$ is empty. Select the largest element of each, and let these be $F_t$ and $F_{t'}$, respectively. Note that $F_t \ne F_{t'}$. Without loss of generaltiy, assume $F_t < F_{t'}$. Then, by (34.1),

$$\sum_{t\in T} t < F_{t+1} \le F_{t'} \le \sum_{t'\in T'} t'.$$

But $\sum_{t\in T} t = \sum_{t'\in T'} t'$, a contradiction, and so, $T = T' = \varnothing$ and $S = S'$, and hence uniqueness.

This concludes the proof. □

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[C. G. Lekkerkerker, *Simon Stevin* **29** (1952), 190–195; section 7.2.1.7; exercise 5.4.2-10; section 7.1.3]

**35.** [*M24*] (*A phi number system.*) Consider real numbers written with the digits 0 and 1 using base $\phi$; thus $(100.1)_\phi = \phi^2 + \phi^{-1}$. Show that there are infinitely many ways to represent the number 1; for example, $1 = (.11)_\phi = (.011111\ldots)_\phi$. But if we require that no two adjacent 1s occur and that the representation does not end with the infinite sequence 01010101..., then every nonnegative number has a unique representation. What are the representations of integers?

In the *phi number system*, there are infinitely many ways to represent the number 1. To see why, note that since $\phi^k = \phi^{k-1} + \phi^{k-2}$,

$$1 = \phi^0 = 1_\phi.$$

We may continue to expand the last term for infinitely many ways to represent the number 1. As

$$1 = \phi^0 = \phi^{-1} + \phi^{-2} = .11_\phi,$$

$$1 = \phi^{-1} + \phi^{-2} = \phi^{-1} + \phi^{-3} + \phi^{-4} = .1011_\phi,$$

or

$$1 = \phi^{-1} + \phi^{-3} + \phi^{-4} = \phi^{-1} + \phi^{-3} + \phi^{-5} + \phi^{-6} = .101011_\phi,$$

ad infinitum. But we may avoid this by requiring that no two adjacent 1s occur and that the representation does not end with the infinite sequence 01010101.... That is, by requiring that all adjacent $\phi^{k-1} + \phi^{k-2}$ terms be instead represented by their sum $\phi^k$ and not avoided by further, infinite expansion of the last term.

The representations of nonnegative integers are then as follows.

**Algorithm 35.1** (*Representation of nonnegative integers in a phi number system.*). Given a nonnegative integer $n$, find its unique representation in the phi number system.

**35.1.a.** [Initialize.] Set $x \leftarrow n$, $D \leftarrow \varnothing$, the set of integer phi exponents.

**35.1.b.** [Test for zero.] If $x = 0$, the algorithm terminates; we have the representation of $n$ by the integer phi exponents in $D$, empty if $n$ zero.

**35.1.c.** [Find largest exponent.] If $x > 0$, find the largest $k$ such that $\phi^k \leq x$, set $D \leftarrow D \cup \{k\}$, $x \leftarrow x - \phi^k$, and return to step 35.1.b. ■

For example, if $n = 0$, $D = \varnothing$ and $n = 0_\phi$; if $n = 1$, $D = \{0\}$ and $n = 1_\phi$; if $n = 2$, $D = \{1, -2\}$ and $n = 10.01_\phi$; etc. Since we always choose $\phi^k$ over any of the terms of the sum $\phi^{k-1} + \phi^{k-2}$, we satisfy the requirement of having no two adjacent 1s and not ending with the infinite sequence 01010101...

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[G. M. Bergman, *Mathematics Magazine* **31** (1957), 98–110]

▶ **36.** [*M32*] (*Fibonacci strings.*) Let $S_1 =$ "a", $S_2 =$ "b", and $S_{n+2} = S_{n+1}S_n$, $n > 0$; in other words, $S_{n+2}$ is formed by placing $S_n$ at the right of $S_{n+1}$. We have $S_3 =$ "ba", $S_4 =$ "bab", $S_5 =$ "babba", etc. Clearly $S_n$ has $F_n$ letters. Explore the properties of $S_n$. (Where do double letters occur? Can you predict the value of the $k$th letter of $S_n$? What is the density of the $b$s? And so on.)

As noted, $S_n$ has $F_n$ letters.

Except for $S_1 = \texttt{a}$, no $S_n$ starts with $\texttt{a}$, but all with $\texttt{b}$; and since $S_2 = \texttt{b}$, every $\texttt{a}$ is preceded by a $\texttt{b}$. The letter $\texttt{b}$ is doubled only when two terms are concatenated. That is, there are no $\texttt{a}$ doubles, only $\texttt{b}$ doubles.

The $k$th letter of $S_n$ is $\alpha(S_n, k)$ where

$$\alpha(S_n, k) = \begin{cases} \texttt{b} & \text{if } n > 1 \text{ and } \lfloor (k+1)\phi^{-1} \rfloor - \lfloor k\phi^{-1} \rfloor = 1 \\ \texttt{a} & \text{otherwise,} \end{cases}$$

for $n > 0$, $1 \leq k \leq F_n$, as proven next.

In the case that $n = 1$, $k = 1$, $\alpha(S_1, 1) = \alpha(\texttt{a}, 1) = \texttt{a}$, and $n \not> 1$. In the case that $n = 2$, $k = 1$, $\alpha(S_2, 1) = \alpha(\texttt{b}, 1) = \texttt{b}$, and

$$\begin{aligned} \lfloor (1+1)\phi^{-1} \rfloor - \lfloor 1 \cdot \phi^{-1} \rfloor &= \lfloor 2 \cdot \phi^{-1} \rfloor - \lfloor 1 \cdot \phi^{-1} \rfloor \\ &= 1 - 0 \\ &= 1. \end{aligned}$$

In the case that $n = 3$, $1 \leq k \leq F_3 = 2$, $\alpha(S_3, k) = \alpha(\texttt{ba}, k)$; and if $k = 1$, then $\alpha(\texttt{ba}, 1) = \texttt{b}$ and

$$\begin{aligned} \lfloor (1+1)\phi^{-1} \rfloor - \lfloor 1 \cdot \phi^{-1} \rfloor &= \lfloor 2 \cdot \phi^{-1} \rfloor - \lfloor 1 \cdot \phi^{-1} \rfloor \\ &= 1 - 0 \\ &= 1; \end{aligned}$$

and if $k = 2$, then $\alpha(\texttt{ba}, 2) = \texttt{a}$ and

$$\begin{aligned} \lfloor (2+1)\phi^{-1} \rfloor - \lfloor 2 \cdot \phi^{-1} \rfloor &= \lfloor 3 \cdot \phi^{-1} \rfloor - \lfloor 2 \cdot \phi^{-1} \rfloor \\ &= 1 - 1 \\ &= 0. \end{aligned}$$

Then, assuming the definition of $\alpha$ holds for $S_{n+1}$, we must show that it holds for $S_{n+1}$. But

$$\alpha(S_{n+1}, k) = \alpha(S_n S_{n-1}, k).$$

In the case that $1 \leq k \leq F_n$, then $\alpha(S_n S_{n-1}, k) = \alpha(S_n, k)$, and $\alpha$ holds by the inductive hypothesis. Otherwise, in the case that $F_n + 1 \leq k \leq F_{n+1}$, then $\alpha(S_n S_{n-1}, k) = \alpha(S_{n-1}, k - F_n) = \alpha(S_{n-1}, k')$ for $1 \leq k' = k - F_n \leq F_{n-1}$, and $\alpha$ holds again by the inductive hypothesis, and hence the result.

The density of the bs is is $\beta(S_n)$ where

$$\beta(S_n) = \begin{cases} \lfloor (F_n + 1)\phi^{-1} \rfloor & \text{if } n > 1 \\ 0 & \text{otherwise,} \end{cases}$$

for $n > 0$, as proven next.

In the case that $n = 1$, $\beta(S_1) = \beta(\mathtt{a}) = 0$, and $n \not> 1$. In the case that $n = 2$, $\beta(S_2) = \beta(\mathtt{b}) = 1$, and

$$\begin{aligned} \lfloor (F_2 + 1)\phi^{-1} \rfloor &= \lfloor (1 + 1)\phi^{-1} \rfloor \\ &= \lfloor 2 \cdot \phi^{-1} \rfloor \\ &= 1. \end{aligned}$$

In the case that $n = 3$, $\beta(S_3) = \beta(\mathtt{ba}) = 1$, and

$$\begin{aligned} \lfloor (F_3 + 1)\phi^{-1} \rfloor &= \lfloor (2 + 1)\phi^{-1} \rfloor \\ &= \lfloor 3 \cdot \phi^{-1} \rfloor \\ &= 1. \end{aligned}$$

Then, assuming the definition of $\beta$ holds for $S_{n+1}$, we must show that it holds for $S_{n+1}$. But

$$\beta(S_{n+1}) = \beta(S_n) + \beta(S_{n-1}).$$

For either term, $\beta$ holds by the inductive hypothesis, and hence the result.

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[K. B. Stolarsky, *Canadian Math. Bull.* **19** (1976), 473–482]

▶ **37.** [*M35*] (R. E. Gaskell, M. J. Whinihan.) Two players compete in the following game: There is a pile containing $n$ chips; the first player removes any number of chips except that he cannot take the whole pile. From then on, the players alternate moves, each person removing one or more chips but *not more than twice as many chips as the preceding player has taken.* The player who removes the last chip wins. (For example, suppose that $n = 11$; player $A$ removes 3 chips; player B may remove up to 6 chips, and he takes 1. There remain 7 chips; player $A$ may take 1 or 2 chips, and he takes 2; player $B$ may remove up to 4, and he picks up 1. There remain 4 chips; player $A$ now takes 1; player $B$ must take at least one chip and player $A$ wins in the following turn.)

What is the best move for the first player to make if there are initially 1000 chips?

The best move for the first player to make if there are initially 1000 chips is to take 13 chips, as explained below.

*Definitions.* Define the game as follows. Let $n_\kappa$ be the number of chips on the $\kappa$th move, $1 \leq \kappa$, $n_\kappa \geq 0$, so that $n_1$ represents the number of chips started with. Let $t_\kappa$ be the number of chips taken in the $\kappa$th move, so that

$$n_\kappa = n_{\kappa-1} - t_{\kappa-1}$$

for $\kappa > 1$. In addition, the rules require that

$$1 \leq t_\kappa \leq q_\kappa = \begin{cases} n_1 - 1 & \text{if } \kappa = 1 \\ 2t_{\kappa-1} & \text{otherwise} \end{cases}$$

for $\kappa \geq 1$, thus $n_1 > 1$ necessarily. The game is won on the $\kappa$th move when finally $n_{\kappa+1} = 0$. We want to find the winning move(s) for the first player, for $\kappa$ odd.

Let

$$n_\kappa = \sum_{1 \leq j \leq r_\kappa} F_{k_{\kappa,j}}$$

be the unique Fibonacci representation of $n_\kappa$, $k_{\kappa,j} > k_{\kappa,j+1} + 1$ for $1 \leq j < r_\kappa$ and $k_{\kappa,r_\kappa} > 1$; and

$$\mu(n_\kappa) = F_{k_{\kappa,r_\kappa}}.$$

The winning move, if it exists, is to remove $t_\kappa \in T_\kappa$ chips where

$$T_\kappa = \left\{ 1 \leq \sum_{j_1 \leq j \leq r_\kappa} F_{k_{\kappa,j}} \leq q_\kappa \;\middle|\; j_1 = 1 \vee F_{k_{\kappa,j_1-1}} > 2 \sum_{j_1 \leq j \leq r_\kappa} F_{k_{\kappa,j}} \right\},$$

where $1 \leq j_1 \leq r_\kappa$.

*Preliminary Result 37.1.* Since $k_{\kappa,r_\kappa} > 1$,

$$\begin{aligned}
F_{k_{\kappa,r_\kappa}+1} > F_{k_{\kappa,r_\kappa}} \\
&\implies F_{k_{\kappa,r_\kappa}+1} + F_{k_{\kappa,r_\kappa}} > F_{k_{\kappa,r_\kappa}} + F_{k_{\kappa,r_\kappa}} \\
&\implies F_{k_{\kappa,r_\kappa}+2} > 2F_{k_{\kappa,r_\kappa}} \\
&\implies F_{k_{\kappa,r_\kappa}+2} > 2\mu\left(\sum_{1 \leq j \leq r_\kappa} F_{k_{\kappa,j}}\right) \\
&\implies F_{k_{\kappa,r_\kappa}+2} > 2\mu(n_\kappa)
\end{aligned}$$

and since $k_{\kappa,r_\kappa} > k_{\kappa,r_\kappa+1} + 1$,

$$\begin{aligned}
F_{k_{\kappa,r_\kappa}} > F_{k_{\kappa,r_\kappa+1}+1} &\implies F_{k_{\kappa,r_\kappa-1}+1} > F_{k_{\kappa,r_\kappa}+2} \\
&\implies F_{k_{\kappa,r_\kappa-1}} \geq F_{k_{\kappa,r_\kappa}+2}
\end{aligned}$$

so that

$$\begin{aligned}
2\mu(n_\kappa) &< F_{k_{\kappa,r_\kappa}+2} \\
&\leq F_{k_{\kappa,r_\kappa-1}} \\
&= \mu\left(\sum_{1 \leq j \leq r_\kappa - 1} F_{k_{\kappa,j}}\right) \\
&= \mu\left(\sum_{1 \leq j \leq r_\kappa} F_{k_{\kappa,j}} - F_{k_{\kappa,r_\kappa}}\right) \\
&= \mu\left(\sum_{1 \leq j \leq r_\kappa} F_{k_{\kappa,j}} - \mu\left(\sum_{1 \leq j \leq r_\kappa} F_{k_{\kappa,j}}\right)\right) \\
&= \mu(n_\kappa - \mu(n_\kappa)).
\end{aligned}$$

That is,

$$\mu(n_\kappa - \mu(n_\kappa)) > 2\mu(n_\kappa). \tag{37.1}$$

*Preliminary Result 37.2.* First note that for $j > 1$ arbitrary, since $F_j = F_{j-1} + F_{j-2}$ and $F_{j-2} \leq F_{j-1}$, we have that

$$F_j \leq F_{j-1} + F_{j-1} = 2F_{j-1}$$

or equivalently that

$$\begin{aligned} F_j \le 2F_{j-1} \quad &\iff \quad 2F_{j-1} \ge F_j \\ &\iff \quad F_{j-1} \ge \frac{1}{2}F_j \\ &\iff \quad -F_{j-1} \le -\frac{1}{2}F_j. \end{aligned}$$

Also note that for $k > j > 0$ arbitrary, if $k = 2u+1$ and $j = 2v+1$ so that $(k-1-j) \bmod 2 = ((2u+1)-1-(2v+1)) \bmod 2 = (2(u-v)-1) \bmod 2 = 1$ and $k \bmod 2 = 1$,

$$\sum_{v+(k-1-j) \bmod 2 \le i \le u} F_{2i-1+k \bmod 2} = F_k - F_{j-1+(k-1-j) \bmod 2}.$$

If $u = v+1 \Longrightarrow k = j+2$,

$$\begin{aligned} \sum_{v+(k-1-j) \bmod 2 \le i \le u} F_{2i-1+k \bmod 2} &= \sum_{v+1 \le i \le u} F_{2i-1+k \bmod 2} \\ &= \sum_{v+1 \le i \le u} F_{2i} \\ &= \sum_{u \le i \le u} F_{2i} \\ &= F_{2u} \\ &= F_{2u+1} - F_{2u-1} \\ &= F_k - F_{2(v+1)-1} \\ &= F_k - F_{2v+2-1} \\ &= F_k - F_{2v+1} \\ &= F_k - F_j \\ &= F_k - F_{j-1+1} \\ &= F_k - F_{j-1+1} \\ &= F_k - F_{j-1+(k-1-j) \bmod 2}. \end{aligned}$$

Assuming

$$\sum_{v+(k-1-j) \bmod 2 \le i \le u} F_{2i-1+k \bmod 2} = F_k - F_{j-1+(k-1-j) \bmod 2},$$

we must show that

$$\sum_{v+(k+2-1-j) \bmod 2 \le i \le u+1} F_{2i-1+(k+2) \bmod 2} = F_{k+2} - F_{j-1+(k+2-1-j) \bmod 2}.$$

But since $(k+2-1-j) \bmod 2 = 1$ and $(k+2) \bmod 2 = 1$,

$$\begin{aligned}
\sum_{v+(k+2-1-j) \bmod 2 \le i \le u+1} F_{2i-1+(k+2) \bmod 2} &= \sum_{v+1 \le i \le u+1} F_{2i-1+(k+2) \bmod 2} \\
&= \sum_{v+1 \le i \le u+1} F_{2i} \\
&= F_{2(u+1)} + \sum_{v+1 \le i \le u} F_{2i} \\
&= F_{2(u+1)} + \sum_{v+1 \le i \le u} F_{2i-1+k \bmod 2} \\
&= F_{2(u+1)} + F_k - F_{j-1+(k-1-j) \bmod 2} \\
&= F_{2u+2} + F_k - F_{j-1+(k-1-j) \bmod 2} \\
&= F_{k+1} + F_k - F_{j-1+(k-1-j) \bmod 2} \\
&= F_{k+2} - F_{j-1+(k-1-j) \bmod 2} \\
&= F_{k+2} - F_{j-1+(k+2-1-j) \bmod 2}
\end{aligned}$$

and hence the result for $k = 2u+1$ and $j = 2v+1$.

If $k = 2u$ and $j = 2v$ so that $(k-1-j) \bmod 2 = (2u-1-2v) \bmod 2 = (2(u-v)-1) \bmod 2 = 1$ and $k \bmod 2 = 0$,

$$\sum_{v+(k-1-j) \bmod 2 \le i \le u} F_{2i-1+k \bmod 2} = F_k - F_{j-1+(k-1-j) \bmod 2}.$$

If $u = v+1 \Longrightarrow k = j+2$,

$$\begin{aligned}
\sum_{v+(k-1-j) \bmod 2 \le i \le u} F_{2i-1+k \bmod 2} &= \sum_{v+1 \le i \le u} F_{2i-1+k \bmod 2} \\
&= \sum_{v+1 \le i \le u} F_{2i-1} \\
&= \sum_{u \le i \le u} F_{2i-1} \\
&= F_{2u-1} \\
&= F_{2u} - F_{2u-2} \\
&= F_k - F_{2(v+1)-2} \\
&= F_k - F_{2v+2-2} \\
&= F_k - F_{2v} \\
&= F_k - F_j \\
&= F_k - F_{j-1+1} \\
&= F_k - F_{j-1+1} \\
&= F_k - F_{j-1+(k-1-j) \bmod 2}.
\end{aligned}$$

Assuming

$$\sum_{v+(k-1-j) \bmod 2 \le i \le u} F_{2i-1+k \bmod 2} = F_k - F_{j-1+(k-1-j) \bmod 2},$$

we must show that

$$\sum_{v+(k+2-1-j) \bmod 2 \le i \le u+1} F_{2i-1+(k+2) \bmod 2} = F_{k+2} - F_{j-1+(k+2-1-j) \bmod 2}.$$

But since $(k+2-1-j) \bmod 2 = 1$ and $(k+2) \bmod 2 = 0$,

$$\begin{aligned}\sum_{v+(k+2-1-j) \bmod 2 \le i \le u+1} F_{2i-1+(k+2) \bmod 2} &= \sum_{v+1 \le i \le u+1} F_{2i-1+(k+2) \bmod 2} \\ &= \sum_{v+1 \le i \le u+1} F_{2i-1} \\ &= F_{2(u+1)-1} + \sum_{v+1 \le i \le u} F_{2i-1} \\ &= F_{2(u+1)-1} + \sum_{v+1 \le i \le u} F_{2i-1+k \bmod 2} \\ &= F_{2(u+1)-1} + F_k - F_{j-1+(k-1-j) \bmod 2} \\ &= F_{2u+2-1} + F_k - F_{j-1+(k-1-j) \bmod 2} \\ &= F_{2u+1} + F_k - F_{j-1+(k-1-j) \bmod 2} \\ &= F_{k+1} + F_k - F_{j-1+(k-1-j) \bmod 2} \\ &= F_{k+2} - F_{j-1+(k-1-j) \bmod 2} \\ &= F_{k+2} - F_{j-1+(k+2-1-j) \bmod 2}\end{aligned}$$

and hence the result for $k = 2u$ and $j = 2v$.

If $k = 2u+1$ and $j = 2v$ so that $(k-1-j) \bmod 2 = (2u+1-1-2v) \bmod 2 = 2(u-v) \bmod 2 = 0$ and $k \bmod 2 = 1$,

$$\sum_{v+(k-1-j) \bmod 2 \le i \le u} F_{2i+1+k \bmod 2} = F_k - F_{j-1+(k-1-j) \bmod 2}.$$

If $u = v \Longrightarrow k = j+1$,

$$\begin{aligned}\sum_{v+(k-1-j) \bmod 2 \le i \le u} F_{2i-1+k \bmod 2} &= \sum_{v+0 \le i \le u} F_{2i-1+k \bmod 2} \\ &= \sum_{v \le i \le u} F_{2i} \\ &= \sum_{u \le i \le u} F_{2i} \\ &= F_{2u} \qquad\qquad = F_{2u+1} - F_{2u-1} \\ &= F_k - F_{2u-1} \\ &= F_k - F_{2v-1} \\ &= F_k - F_{j-1} \\ &= F_k - F_{j-1+0} \\ &= F_k - F_{j-1+(k-1-j) \bmod 2}.\end{aligned}$$

Assuming

$$\sum_{v+(k-1-j) \bmod 2 \le i \le u} F_{2i-1+k \bmod 2} = F_k - F_{j-1+(k-1-j) \bmod 2},$$

we must show that

$$\sum_{v+(k+2-1-j) \bmod 2 \le i \le u+1} F_{2i-1+(k+2) \bmod 2} = F_{k+2} - F_{j-1+(k+2-1-j) \bmod 2}.$$

But since $(k+2-1-j) \bmod 2 = 0$ and $(k+2) \bmod 2 = 1$,

$$\begin{aligned}
\sum_{v+(k-1-j) \bmod 2 \le i \le u+1} F_{2i-1+(k+2) \bmod 2} &= \sum_{v+0 \le i \le u+1} F_{2i-1+(k+2) \bmod 2} \\
&= \sum_{v \le i \le u+1} F_{2i} \\
&= F_{2(u+1)} + \sum_{v \le i \le u} F_{2i} \\
&= F_{2u+2} + \sum_{v+(k-1-j) \bmod 2 \le i \le u} F_{2i-1+k \bmod 2} \\
&= F_{2u+2} + F_k - F_{j-1+(k-1-j) \bmod 2} \\
&= F_{k+1} + F_k - F_{j-1+(k-1-j) \bmod 2} \\
&= F_{k+2} - F_{j-1+(k-1-j) \bmod 2} \\
&= F_{k+2} - F_{j-1+(k+2-1-j) \bmod 2}
\end{aligned}$$

and hence the result for $k = 2u+1$ and $j = 2v$.

If $k = 2u$ and $j = 2v-1$ so that $(k-1-j) \bmod 2 = (2u-1-(2v-1)) \bmod 2 = 2(u-v) \bmod 2 = 0$ and $k \bmod 2 = 0$,

$$\sum_{v+(k-1-j) \bmod 2 \le i \le u} F_{2i+1+k \bmod 2} = F_k - F_{j-1+(k-1-j) \bmod 2}.$$

If $u = v \Longrightarrow k = j+1$,

$$\begin{aligned}
\sum_{v+(k-1-j) \bmod 2 \le i \le u} F_{2i-1+k \bmod 2} &= \sum_{v+0 \le i \le u} F_{2i-1+k \bmod 2} \\
&= \sum_{v \le i \le u} F_{2i-1} \\
&= \sum_{u \le i \le u} F_{2i-1} \\
&= F_{2u-1} \qquad\qquad = F_{2u} - F_{2u-2} \\
&= F_k - F_{2v-2} \\
&= F_k - F_{j-1} \\
&= F_k - F_{j-1+0} \\
&= F_k - F_{j-1+(k-1-j) \bmod 2}.
\end{aligned}$$

Assuming

$$\sum_{v+(k-1-j) \bmod 2 \le i \le u} F_{2i-1+k \bmod 2} = F_k - F_{j-1+(k-1-j) \bmod 2},$$

we must show that

$$\sum_{v+(k+2-1-j) \bmod 2 \le i \le u+1} F_{2i-1+(k+2) \bmod 2} = F_{k+2} - F_{j-1+(k+2-1-j) \bmod 2}.$$

But since $(k+2-1-j) \bmod 2 = 0$ and $(k+2) \bmod 2 = 0$,

$$\begin{aligned}
\sum_{v+(k-1-j) \bmod 2 \le i \le u+1} F_{2i-1+(k+2) \bmod 2} &= \sum_{v+0 \le i \le u+1} F_{2i-1+(k+2) \bmod 2} \\
&= \sum_{v \le i \le u+1} F_{2i-1} \\
&= F_{2(u+1)-1} + \sum_{v \le i \le u} F_{2i-1} \\
&= F_{2u+2-1} + \sum_{v+(k-1-j) \bmod 2 \le i \le u} F_{2i-1+k \bmod 2} \\
&= F_{2u+1} + F_k - F_{j-1+(k-1-j) \bmod 2} \\
&= F_{k+1} + F_k - F_{j-1+(k-1-j) \bmod 2} \\
&= F_{k+2} - F_{j-1+(k-1-j) \bmod 2} \\
&= F_{k+2} - F_{j-1+(k+2-1-j) \bmod 2}
\end{aligned}$$

and hence the result for $k = 2u$ and $j = 2v - 1$.

Hence, in any and all cases

$$\begin{aligned}
&\sum_{v+(k-1-j) \bmod 2 \le i \le u} F_{2i+1+k \bmod 2} \\
&= \sum_{\lfloor j/2 \rfloor + (k-1-j) \bmod 2 \le i \le \lfloor k/2 \rfloor} F_{2i+1+k \bmod 2} \\
&= \sum_{2\lfloor j/2 \rfloor + 2((k-1-j) \bmod 2) + 1 + k \bmod 2 \le i \le 2\lfloor k/2 \rfloor + 1 + k \bmod 2} F_i \\
&= F_k - F_{j-1+(k-1-j) \bmod 2}.
\end{aligned}$$

Also, if $k = 2u$, so that $k \bmod 2 = 0$,

$$\begin{aligned}
F_k &= F_{2u} \\
&= \sum_{0 \le i \le u-1} F_{2i+1} \\
&= \sum_{1 \le i \le u} F_{2i-1} \\
&= \sum_{1 \le i \le u} (F_{2i+1} - F_{2i}) \\
&= \sum_{1 \le i \le u} F_{2i+1} - \sum_{1 \le i \le u} F_{2i} \\
&= \sum_{1 \le i \le u} F_{2i+1} - \sum_{0 \le i \le u-1} F_{2i+2} \\
&\le \sum_{1 \le i \le u} F_{2i+1} - \sum_{1 \le i \le u-1} F_{2i+1} \\
&\le \sum_{1 \le i \le u} F_{2i+1} - \sum_{1 \le i \le v+(k-1-j) \bmod 2-1} F_{2i+1} \\
&= \sum_{v+(k-1-j) \bmod 2 \le i \le u} F_{2i+1} \\
&= \sum_{v+(k-1-j) \bmod 2 \le i \le u} F_{2i+1+k \bmod 2},
\end{aligned}$$

,

and if $k = 2u + 1$, so that $k \bmod 2 = 1$,

$$\begin{aligned}
F_k &= F_{2u+1} \\
&= 1 + \sum_{1 \le i \le u} F_{2i} \\
&= 1 + \sum_{1 \le i \le u} (F_{2i+1} - F_{2i-1}) \\
&= 1 + \sum_{1 \le i \le u} F_{2i+1} - \sum_{1 \le i \le u} F_{2i-1} \\
&= 1 + \sum_{1 \le i \le u} F_{2i+1} - \sum_{0 \le i \le u-1} F_{2i+1} \\
&= \sum_{1 \le i \le u} F_{2i+1} - \sum_{1 \le i \le u-1} F_{2i+1} \\
&\le \sum_{1 \le i \le u} F_{2i+1} - \sum_{1 \le i \le v+(k-1-j) \bmod 2-1} F_{2i+1} \\
&= \sum_{v+(k-1-j) \bmod 2 \le i \le u} F_{2i+1} \\
&\le \sum_{v+(k-1-j) \bmod 2 \le i \le u} F_{2i+1+k \bmod 2},
\end{aligned}$$

,

so that in either case,

$$\begin{aligned} F_k &\le \sum_{v+(k-1-j) \bmod 2 \le i \le u} F_{2i+1+k \bmod 2} \\ &= \sum_{2\lfloor j/2\rfloor+2((k-1-j) \bmod 2)+1+k \bmod 2 \le i \le 2\lfloor k/2\rfloor+1+k \bmod 2} F_i. \end{aligned}$$

Then let $\mu(m) = F_j$, so that if $m < F_k$,

$$\begin{aligned} m &< F_k \\ &\le \sum_{2\lfloor j/2\rfloor+2((k-1-j) \bmod 2)+1+k \bmod 2 \le i \le 2\lfloor k/2\rfloor+1+k \bmod 2} F_i \\ &= -F_{j-1+(k-1-j) \bmod 2} + F_k \end{aligned}$$

In the case that $(k-1-j) \bmod 2 = 0$,

$$\begin{aligned} -F_{j-1+(k-1-j) \bmod 2} &= -F_{j-1+0} \\ &= -F_{j-1} \\ &\le -\frac{1}{2}F_j; \end{aligned}$$

and in the case that $(k-1-j) \bmod 2 = 1$,

$$\begin{aligned} -F_{j-1+(k-1-j) \bmod 2} &= -F_{j-1+1} \\ &= -F_j \\ &\le -\frac{1}{2}F_{j+1} \\ &\le -\frac{1}{2}F_j. \end{aligned}$$

And so, in either case,

$$\begin{aligned} -F_{j-1+(k-1-j) \bmod 2} + F_k & \\ &\le -\frac{1}{2}F_j + F_k \\ &= -\frac{1}{2}\mu(m) + F_k \end{aligned}$$

if and only if

$$\begin{aligned} m \le -\frac{1}{2}\mu(m) + F_k \quad &\iff \quad -F_k + m \le -\frac{1}{2}\mu(m) \\ &\iff \quad F_k - m \ge \frac{1}{2}\mu(m) \\ &\iff \quad 2(F_k - m) \ge \mu(m). \end{aligned}$$

That is,

$$\mu(m) \le 2(F_k - m) \tag{37.2}$$

for $1 \le m < F_k$.

*Preliminary Result 37.3.* Since $F_{\kappa,r_\kappa} = \mu(n_\kappa) > m \geq 1$, by (37.2)

$$\begin{aligned} 2(F_{\kappa,r_\kappa} - m) &= 2(\mu(n_\kappa) - m) \\ &\geq \mu(m) \\ &= \mu\left(\sum_{1 \leq j \leq r_\kappa - 1} F_{k_{\kappa,j}} + m\right) \\ &= \mu(n_\kappa - F_{k_{\kappa,r_\kappa}} + m) \\ &= \mu(n_\kappa - \mu(n_\kappa) + m). \end{aligned}$$

That is,

$$\mu(n_\kappa - \mu(n_\kappa) + m) \leq 2(\mu(n_\kappa) - m) \tag{37.3}$$

for $1 \leq m < \mu(n_\kappa)$.

*Preliminary Result 37.4.* By (37.3), with $m = \mu(n_\kappa) - m'$,

$$\begin{aligned} \mu(n_\kappa - \mu(n_\kappa) + \mu(n_\kappa) - m') &= \mu(n_\kappa - m') \\ &\leq 2(\mu(n_\kappa) - \mu(n_\kappa) - m') \\ &= 2m'. \end{aligned}$$

That is,

$$\mu(n_\kappa - m) \leq 2m \tag{37.4}$$

for $1 \leq m < \mu(n_\kappa)$.

*Proof.* In the case that $\mu(n_\kappa) \leq q_\kappa$ and $q_\kappa \geq n_\kappa$, we may win immediately in the $\kappa$th move by taking $t_\kappa = n_\kappa$ chips. But since $n_\kappa \geq 1$,

$$q_\kappa \geq n_\kappa \geq 1 \quad \Longleftrightarrow \quad 1 \leq \sum_{j_1 \leq j \leq r_\kappa} F_{k_{\kappa,j}} \leq q_\kappa$$

with $j_1 = 1$, so that $n_\kappa \in T_\kappa$.

Otherwise, if $\mu(n_\kappa) \leq q_\kappa$ but $q_\kappa < n_\kappa$, we may take $t_\kappa = \mu(n_\kappa)$ chips in order to leave the other player in an unwinnable state where $\mu(n_{\kappa+1}) > q_{\kappa+1}$. The move is valid since $\mu(n_\kappa) \geq 1$,

$$q_\kappa \geq \mu(n_\kappa) \geq 1 \quad \Longleftrightarrow \quad 1 \leq \sum_{j_1 \leq j \leq r_\kappa} F_{k_{\kappa,j}} \leq q_\kappa$$

with $j_1 = r_\kappa$, so that $\mu(n_\kappa) \in T_\kappa$, since by (37.1),

$$\begin{aligned} 2\sum_{j_1 \leq j \leq r_\kappa} F_{k_{\kappa,j}} &= 2\mu(n_\kappa) \\ &< \mu(n_\kappa - \mu(n_\kappa)) \\ &= F_{k_{\kappa,r_\kappa - 1}}. \end{aligned}$$

The next state (to be shown to be unwinnable further below) is indeed one where $\mu(n_{\kappa+1}) > q_{\kappa+1}$, since also by (37.1),

$$\begin{aligned} &\mu(n_{\kappa+1}) \\ &\quad = \mu(n_\kappa - t_\kappa) \\ &\quad = \mu(n_\kappa - \mu(n_\kappa)) \\ &\quad > 2\mu(n_\kappa) \\ &\quad = 2t_\kappa \\ &\quad = q_{\kappa+1}. \end{aligned}$$

In the case that $\mu(n_\kappa) > q_\kappa$, there is no winnable move since $q_\kappa < n_\kappa$; but *any* move $t_\kappa$ will lead to a winnable state for the next player with $\mu(n_{\kappa+1}) \le q_{\kappa+1}$. This follows from (37.4), since $1 \le t_\kappa < \mu(n_\kappa)$ and

$$\begin{aligned} \mu(n_{\kappa+1}) \\ &= \mu(n_\kappa - t_\kappa) \\ &\le 2t_\kappa \\ &= q_{\kappa+1}. \end{aligned}$$

*Example.* Here we explain how we determined that taking 13 chips is the only winning move for the first player to make if there are initially 1000 chips. In this example,

$$n_1 = 1000 = F_{k_{1,1}} + F_{k_{1,r_1}} = F_{k_{1,1}} + F_{k_{1,2}} = F_{16} + F_7 = 987 + 13,$$

and $\kappa = 1$, so that $q = 1000 - 1 = 999$. For $j_1 = r_1$, since

$$987 = F_{k_{1,1}} = F_{k_{1,2-1}} = F_{k_{1,r_1-1}} > 2 \sum_{r_1 \le j \le r_1} F_{k_{1,j}} = 2F_{k_{1,r_1}} = 2 \cdot 13 = 26,$$

we have a single winning move

$$t_1 \in T_1 = \{13\},$$

hence the unique solution.

---

[M. J. Whinihan, *Fibonacci Quart.* **1** (December 1963), 9–12; A. Schwenk, *Fibonacci Quarterly* **8** (1970), 225–234]

**38.** [*35*] Write a computer program that plays the game described in the previous exercise and that plays optimally.

The following Java code plays the game described in exercise 37, and plays optimally.

```
class Options {

   public Options(String[] arguments) throws NumberFormatException {
      for (int index = 0; index < arguments.length; ++index) {
         switch (arguments[index]) {
            case "-n":
               if ((numberOfChips = Integer.parseInt(arguments[++index])) <= 1) {
                  throw new IllegalArgumentException(
                     String.format(
                        "number␣of␣chips␣n␣must␣be␣n␣>␣1:␣%d",
                        numberOfChips
                     )
                  );
               }
               break;
            case "-p":
               isUserFirst = Integer.parseInt(arguments[++index]) % 2 == 1;
               break;
            default:
               throw new IllegalArgumentException(arguments[index]);
         }
      }

      assert (numberOfChips > 1);

   }

   public     int getNumberOfChips() { return numberOfChips; }
   public boolean isUserFirst()      { return isUserFirst;   }

   public int readNumberOfChipsTaken(InputStream in, int numberOfChipsTakeable) {
      int numberOfChipsTaken = (new Scanner(in)).nextInt();
      if ((numberOfChipsTaken < 1) || (numberOfChipsTaken > numberOfChipsTakeable)) {
         throw new IllegalArgumentException(
            String.format(
               "number␣of␣chips␣taken␣t␣must␣be␣1␣<=␣t␣<=␣%d:␣%d",
```

```
               numberOfChipsTakeable,
               numberOfChipsTaken
            )
         );
      }
      return numberOfChipsTaken;
   }

   private     int numberOfChips = 2;
   private boolean isUserFirst   = true;

}

class State {

   public State(int initialNumberOfChips, boolean isUserFirst) {
      turn = 1;
      isUserTurn = isUserFirst;
      numberOfChips = initialNumberOfChips;
      numberOfChipsTakeable = numberOfChips - 1;
   }

   public void take(int numberOfChipsTaken) {

      assert ((1 <= numberOfChipsTaken) && (numberOfChipsTaken <= numberOfChipsTakeable));

      ++turn;
      isUserTurn = !isUserTurn;
      numberOfChips -= numberOfChipsTaken;
      numberOfChipsTakeable = 2*numberOfChipsTaken;
   }

   public     int getTurn()                  { return turn;                  }
   public boolean isUserTurn()               { return isUserTurn;            }
   public     int getNumberOfChips()         { return numberOfChips;         }
   public     int getNumberOfChipsTakeable() { return numberOfChipsTakeable; }

   public void writePreSummary(PrintStream out) {
      out.printf("Number␣of␣Chips:␣%d%n", numberOfChips);
      out.printf("␣␣␣␣␣Goes␣First:␣%s%n", isUserTurn? "User" : "Computer");
      out.printf("---------------%n");
   }

   public void writeSummary(PrintStream out) {
      out.printf(
         "[State]␣Turn:␣%d␣(%s);␣Chips:␣%d;␣Takeable:␣%d%n",
         turn,
         isUserTurn? "User" : "Computer",
         numberOfChips,
         numberOfChipsTakeable
      );
   }

   public void writePostSummary(PrintStream out) {
      out.printf("------%n");
      out.printf("␣Turns:␣%d%n", turn - 1);
      out.printf("Winner:␣%s%n", !isUserTurn? "User" : "Computer");
   }

   private     int turn;
   private boolean isUserTurn;
   private     int numberOfChips;
   private     int numberOfChipsTakeable;

}

class FibonacciNumber {

   public FibonacciNumber(int index, int value) {
      this.index = index;
      this.value = value;
   }

   public int getIndex() { return index; }
   public int getValue() { return value; }

   private int index;
   private int value;

}

class FibonacciNumbers {

   public FibonacciNumber get(int index) {
```

```
      assert (index >= 0);

      FibonacciNumber fibonacciNumber = fibonacciNumberCache.get(index);
      if (fibonacciNumber == null) {
         fibonacciNumber = calculate(index);
         fibonacciNumberCache.put(index, fibonacciNumber);
      }
      return fibonacciNumber;
   }

   public FibonacciNumber getLargestLessThanOrEqualTo(int value) {
      int index = 0;
      FibonacciNumber fibonacciNumber = get(index++);
      while (true) {
         FibonacciNumber nextFibonacciNumber = get(index++);
         if (nextFibonacciNumber.getValue() > value) {
            break;
         }
         fibonacciNumber = nextFibonacciNumber;
      }
      return fibonacciNumber;
   }

   protected FibonacciNumber calculate(int index) {
      FibonacciNumber fibonacciNumber;
      switch (index) {
         case 0:
         case 1:
            fibonacciNumber = new FibonacciNumber(index, index);
            break;
         default:
            fibonacciNumber = new FibonacciNumber(
               index,
               get(index-1).getValue() + get(index-2).getValue()
            );
            break;
      }
      return fibonacciNumber;
   }

   private Map<Integer,FibonacciNumber> fibonacciNumberCache = new HashMap<>();

}

class FibonacciBaseNumber {

   public FibonacciBaseNumber(FibonacciNumbers fibonacciNumbers, int value) {

      assert (value > 0);

      this.value = value;
      while (value > 0) {
         FibonacciNumber digit = fibonacciNumbers.getLargestLessThanOrEqualTo(value);
         digits.add(digit);
         value -= digit.getValue();
      }
   }

   public int getSum(int fromIndex, int toIndex) {
      int sum = 0;
      for (int index = fromIndex; index <= toIndex; ++index) {
         sum += get(index).getValue();
      }
      return sum;
   }

   public                      int getValue()     { return value;              }
   public                      int size()         { return digits.size();      }
   public          FibonacciNumber get(int index) { return digits.get(index); }
   public Stream<FibonacciNumber> stream()        { return digits.stream();    }

   private                    int value;
   private List<FibonacciNumber> digits = new ArrayList<FibonacciNumber>();

}

class Solver {

   public Solver(FibonacciNumbers fibonacciNumbers, State state) {
      fibonacciBaseNumber = new FibonacciBaseNumber(fibonacciNumbers, state.getNumberOfChips());
      for (int index = 0; index < fibonacciBaseNumber.size(); ++index) {
         int sum = fibonacciBaseNumber.getSum(index, fibonacciBaseNumber.size() - 1);
         if ((1 <= sum) && (sum <= state.getNumberOfChipsTakeable())) {
```

```
            if ((index == 0) || (fibonacciBaseNumber.get(index - 1).getValue() > 2*sum)) {
               numberOfChipsToTake.add(sum);
            }
         }
      }
   }

   public int getOptimalNumberOfChipsToTake() {
      int optimalNumberOfChipsToTake = 1;
      if (numberOfChipsToTake.size() > 0) {
         optimalNumberOfChipsToTake = numberOfChipsToTake.first();
      }
      return optimalNumberOfChipsToTake;
   }

   public void writeSummary(PrintStream out) {
      out.printf(
         "[Solve]␣Chips:␣%d␣=␣%s␣=␣%s%n",
         fibonacciBaseNumber.getValue(),
         String.join("␣+␣", fibonacciBaseNumber.stream()
            .map(f -> String.format("F_%d", f.getIndex())).collect(Collectors.toList())
         ),
         String.join("␣+␣", fibonacciBaseNumber.stream()
               .map(f -> Integer.toString(f.getValue())).collect(Collectors.toList())
         )
      );
      out.printf(
         "[Solve]␣Optimal:␣%d;␣Possible:␣{%s}%n",
         getOptimalNumberOfChipsToTake(),
         String.join(",␣", numberOfChipsToTake.stream()
               .map(t -> Integer.toString(t)).collect(Collectors.toList())
         )
      );
   }

   private FibonacciBaseNumber fibonacciBaseNumber;
   private  SortedSet<Integer> numberOfChipsToTake = new TreeSet<Integer>();

}

Options options = new Options(arguments);
State state = new State(options.getNumberOfChips(), options.isUserFirst());
FibonacciNumbers fibonacciNumbers = new FibonacciNumbers();
state.writePreSummary(System.out);
do {
   Solver solver = new Solver(fibonacciNumbers, state);
   state.writeSummary(System.out);
   solver.writeSummary(System.out);
   int numberOfChipsTaken;
   if (state.isUserTurn()) {
      System.out.printf("[Input]␣User␣Takes:␣");
      numberOfChipsTaken = options.readNumberOfChipsTaken(System.in, state.getNumberOfChipsTakeable());
   } else {
      System.out.printf("[Input]␣Computer␣Takes:␣");
      numberOfChipsTaken = solver.getOptimalNumberOfChipsToTake();
      System.out.printf("%d%n", numberOfChipsTaken);
   }
   state.take(numberOfChipsTaken);
} while (state.getNumberOfChips() > 0);
state.writePostSummary(System.out);
```

Sample output for a game starting with 1000 chips where the computer went first (`-n 1000 -p 2`), lasting for 351 turns, is shown below.

```
Number of Chips: 1000
     Goes First: Computer
---------------
[State] Turn: 1 (Computer); Chips: 1000; Takeable: 999
[Solve] Chips: 1000 = F_16 + F_7 = 987 + 13
[Solve] Optimal: 13; Possible: {13}
[Input] Computer Takes: 13
[State] Turn: 2 (User); Chips: 987; Takeable: 26
[Solve] Chips: 987 = F_16 = 987
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 26
[State] Turn: 3 (Computer); Chips: 961; Takeable: 52
[Solve] Chips: 961 = F_15 + F_13 + F_11 + F_8 + F_6 = 610 + 233 + 89 + 21 + 8
[Solve] Optimal: 8; Possible: {8, 29}
[Input] Computer Takes: 8
[State] Turn: 4 (User); Chips: 953; Takeable: 16
[Solve] Chips: 953 = F_15 + F_13 + F_11 + F_8 = 610 + 233 + 89 + 21
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 16
```

```
[State] Turn: 5 (Computer); Chips: 937; Takeable: 32
[Solve] Chips: 937 = F_15 + F_13 + F_11 + F_5 = 610 + 233 + 89 + 5
[Solve] Optimal: 5; Possible: {5}
[Input] Computer Takes: 5
[State] Turn: 6 (User); Chips: 932; Takeable: 10
[Solve] Chips: 932 = F_15 + F_13 + F_11 = 610 + 233 + 89
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 10
[State] Turn: 7 (Computer); Chips: 922; Takeable: 20
[Solve] Chips: 922 = F_15 + F_13 + F_10 + F_8 + F_4 = 610 + 233 + 55 + 21 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 8 (User); Chips: 919; Takeable: 6
[Solve] Chips: 919 = F_15 + F_13 + F_10 + F_8 = 610 + 233 + 55 + 21
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 9 (Computer); Chips: 913; Takeable: 12
[Solve] Chips: 913 = F_15 + F_13 + F_10 + F_7 + F_3 = 610 + 233 + 55 + 13 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 10 (User); Chips: 911; Takeable: 4
[Solve] Chips: 911 = F_15 + F_13 + F_10 + F_7 = 610 + 233 + 55 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 11 (Computer); Chips: 907; Takeable: 8
[Solve] Chips: 907 = F_15 + F_13 + F_10 + F_6 + F_2 = 610 + 233 + 55 + 8 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 12 (User); Chips: 906; Takeable: 2
[Solve] Chips: 906 = F_15 + F_13 + F_10 + F_6 = 610 + 233 + 55 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 13 (Computer); Chips: 904; Takeable: 4
[Solve] Chips: 904 = F_15 + F_13 + F_10 + F_5 + F_2 = 610 + 233 + 55 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 14 (User); Chips: 903; Takeable: 2
[Solve] Chips: 903 = F_15 + F_13 + F_10 + F_5 = 610 + 233 + 55 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 15 (Computer); Chips: 901; Takeable: 4
[Solve] Chips: 901 = F_15 + F_13 + F_10 + F_4 = 610 + 233 + 55 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 16 (User); Chips: 898; Takeable: 6
[Solve] Chips: 898 = F_15 + F_13 + F_10 = 610 + 233 + 55
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 17 (Computer); Chips: 892; Takeable: 12
[Solve] Chips: 892 = F_15 + F_13 + F_9 + F_7 + F_3 = 610 + 233 + 34 + 13 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 18 (User); Chips: 890; Takeable: 4
[Solve] Chips: 890 = F_15 + F_13 + F_9 + F_7 = 610 + 233 + 34 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 19 (Computer); Chips: 886; Takeable: 8
[Solve] Chips: 886 = F_15 + F_13 + F_9 + F_6 + F_2 = 610 + 233 + 34 + 8 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 20 (User); Chips: 885; Takeable: 2
[Solve] Chips: 885 = F_15 + F_13 + F_9 + F_6 = 610 + 233 + 34 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 21 (Computer); Chips: 883; Takeable: 4
[Solve] Chips: 883 = F_15 + F_13 + F_9 + F_5 + F_2 = 610 + 233 + 34 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 22 (User); Chips: 882; Takeable: 2
[Solve] Chips: 882 = F_15 + F_13 + F_9 + F_5 = 610 + 233 + 34 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 23 (Computer); Chips: 880; Takeable: 4
[Solve] Chips: 880 = F_15 + F_13 + F_9 + F_4 = 610 + 233 + 34 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 24 (User); Chips: 877; Takeable: 6
[Solve] Chips: 877 = F_15 + F_13 + F_9 = 610 + 233 + 34
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 25 (Computer); Chips: 871; Takeable: 12
[Solve] Chips: 871 = F_15 + F_13 + F_8 + F_5 + F_3 = 610 + 233 + 21 + 5 + 2
[Solve] Optimal: 2; Possible: {2, 7}
[Input] Computer Takes: 2
```

```
[State] Turn: 26 (User); Chips: 869; Takeable: 4
[Solve] Chips: 869 = F_15 + F_13 + F_8 + F_5 = 610 + 233 + 21 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 27 (Computer); Chips: 865; Takeable: 8
[Solve] Chips: 865 = F_15 + F_13 + F_8 + F_2 = 610 + 233 + 21 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 28 (User); Chips: 864; Takeable: 2
[Solve] Chips: 864 = F_15 + F_13 + F_8 = 610 + 233 + 21
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 29 (Computer); Chips: 862; Takeable: 4
[Solve] Chips: 862 = F_15 + F_13 + F_7 + F_5 + F_2 = 610 + 233 + 13 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 30 (User); Chips: 861; Takeable: 2
[Solve] Chips: 861 = F_15 + F_13 + F_7 + F_5 = 610 + 233 + 13 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 31 (Computer); Chips: 859; Takeable: 4
[Solve] Chips: 859 = F_15 + F_13 + F_7 + F_4 = 610 + 233 + 13 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 32 (User); Chips: 856; Takeable: 6
[Solve] Chips: 856 = F_15 + F_13 + F_7 = 610 + 233 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 33 (Computer); Chips: 850; Takeable: 12
[Solve] Chips: 850 = F_15 + F_13 + F_5 + F_3 = 610 + 233 + 5 + 2
[Solve] Optimal: 2; Possible: {2, 7}
[Input] Computer Takes: 2
[State] Turn: 34 (User); Chips: 848; Takeable: 4
[Solve] Chips: 848 = F_15 + F_13 + F_5 = 610 + 233 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 35 (Computer); Chips: 844; Takeable: 8
[Solve] Chips: 844 = F_15 + F_13 + F_2 = 610 + 233 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 36 (User); Chips: 843; Takeable: 2
[Solve] Chips: 843 = F_15 + F_13 = 610 + 233
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 37 (Computer); Chips: 841; Takeable: 4
[Solve] Chips: 841 = F_15 + F_12 + F_10 + F_8 + F_6 + F_4 = 610 + 144 + 55 + 21 + 8 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 38 (User); Chips: 838; Takeable: 6
[Solve] Chips: 838 = F_15 + F_12 + F_10 + F_8 + F_6 = 610 + 144 + 55 + 21 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 39 (Computer); Chips: 832; Takeable: 12
[Solve] Chips: 832 = F_15 + F_12 + F_10 + F_8 + F_3 = 610 + 144 + 55 + 21 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 40 (User); Chips: 830; Takeable: 4
[Solve] Chips: 830 = F_15 + F_12 + F_10 + F_8 = 610 + 144 + 55 + 21
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 41 (Computer); Chips: 826; Takeable: 8
[Solve] Chips: 826 = F_15 + F_12 + F_10 + F_7 + F_4 + F_2 = 610 + 144 + 55 + 13 + 3 + 1
[Solve] Optimal: 1; Possible: {1, 4}
[Input] Computer Takes: 1
[State] Turn: 42 (User); Chips: 825; Takeable: 2
[Solve] Chips: 825 = F_15 + F_12 + F_10 + F_7 + F_4 = 610 + 144 + 55 + 13 + 3
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 43 (Computer); Chips: 823; Takeable: 4
[Solve] Chips: 823 = F_15 + F_12 + F_10 + F_7 + F_2 = 610 + 144 + 55 + 13 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 44 (User); Chips: 822; Takeable: 2
[Solve] Chips: 822 = F_15 + F_12 + F_10 + F_7 = 610 + 144 + 55 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 45 (Computer); Chips: 820; Takeable: 4
[Solve] Chips: 820 = F_15 + F_12 + F_10 + F_6 + F_4 = 610 + 144 + 55 + 8 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 46 (User); Chips: 817; Takeable: 6
[Solve] Chips: 817 = F_15 + F_12 + F_10 + F_6 = 610 + 144 + 55 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
```

```
[State] Turn: 47 (Computer); Chips: 811; Takeable: 12
[Solve] Chips: 811 = F_15 + F_12 + F_10 + F_3 = 610 + 144 + 55 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 48 (User); Chips: 809; Takeable: 4
[Solve] Chips: 809 = F_15 + F_12 + F_10 = 610 + 144 + 55
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 49 (Computer); Chips: 805; Takeable: 8
[Solve] Chips: 805 = F_15 + F_12 + F_9 + F_7 + F_4 + F_2 = 610 + 144 + 34 + 13 + 3 + 1
[Solve] Optimal: 1; Possible: {1, 4}
[Input] Computer Takes: 1
[State] Turn: 50 (User); Chips: 804; Takeable: 2
[Solve] Chips: 804 = F_15 + F_12 + F_9 + F_7 + F_4 = 610 + 144 + 34 + 13 + 3
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 51 (Computer); Chips: 802; Takeable: 4
[Solve] Chips: 802 = F_15 + F_12 + F_9 + F_7 + F_2 = 610 + 144 + 34 + 13 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 52 (User); Chips: 801; Takeable: 2
[Solve] Chips: 801 = F_15 + F_12 + F_9 + F_7 = 610 + 144 + 34 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 53 (Computer); Chips: 799; Takeable: 4
[Solve] Chips: 799 = F_15 + F_12 + F_9 + F_6 + F_4 = 610 + 144 + 34 + 8 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 54 (User); Chips: 796; Takeable: 6
[Solve] Chips: 796 = F_15 + F_12 + F_9 + F_6 = 610 + 144 + 34 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 55 (Computer); Chips: 790; Takeable: 12
[Solve] Chips: 790 = F_15 + F_12 + F_9 + F_3 = 610 + 144 + 34 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 56 (User); Chips: 788; Takeable: 4
[Solve] Chips: 788 = F_15 + F_12 + F_9 = 610 + 144 + 34
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 57 (Computer); Chips: 784; Takeable: 8
[Solve] Chips: 784 = F_15 + F_12 + F_8 + F_6 + F_2 = 610 + 144 + 21 + 8 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 58 (User); Chips: 783; Takeable: 2
[Solve] Chips: 783 = F_15 + F_12 + F_8 + F_6 = 610 + 144 + 21 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 59 (Computer); Chips: 781; Takeable: 4
[Solve] Chips: 781 = F_15 + F_12 + F_8 + F_5 + F_2 = 610 + 144 + 21 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 60 (User); Chips: 780; Takeable: 2
[Solve] Chips: 780 = F_15 + F_12 + F_8 + F_5 = 610 + 144 + 21 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 61 (Computer); Chips: 778; Takeable: 4
[Solve] Chips: 778 = F_15 + F_12 + F_8 + F_4 = 610 + 144 + 21 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 62 (User); Chips: 775; Takeable: 6
[Solve] Chips: 775 = F_15 + F_12 + F_8 = 610 + 144 + 21
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 63 (Computer); Chips: 769; Takeable: 12
[Solve] Chips: 769 = F_15 + F_12 + F_7 + F_3 = 610 + 144 + 13 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 64 (User); Chips: 767; Takeable: 4
[Solve] Chips: 767 = F_15 + F_12 + F_7 = 610 + 144 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 65 (Computer); Chips: 763; Takeable: 8
[Solve] Chips: 763 = F_15 + F_12 + F_6 + F_2 = 610 + 144 + 8 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 66 (User); Chips: 762; Takeable: 2
[Solve] Chips: 762 = F_15 + F_12 + F_6 = 610 + 144 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 67 (Computer); Chips: 760; Takeable: 4
[Solve] Chips: 760 = F_15 + F_12 + F_5 + F_2 = 610 + 144 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
```

```
[State] Turn: 68 (User); Chips: 759; Takeable: 2
[Solve] Chips: 759 = F_15 + F_12 + F_5 = 610 + 144 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 69 (Computer); Chips: 757; Takeable: 4
[Solve] Chips: 757 = F_15 + F_12 + F_4 = 610 + 144 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 70 (User); Chips: 754; Takeable: 6
[Solve] Chips: 754 = F_15 + F_12 = 610 + 144
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 71 (Computer); Chips: 748; Takeable: 12
[Solve] Chips: 748 = F_15 + F_11 + F_9 + F_7 + F_3 = 610 + 89 + 34 + 13 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 72 (User); Chips: 746; Takeable: 4
[Solve] Chips: 746 = F_15 + F_11 + F_9 + F_7 = 610 + 89 + 34 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 73 (Computer); Chips: 742; Takeable: 8
[Solve] Chips: 742 = F_15 + F_11 + F_9 + F_6 + F_2 = 610 + 89 + 34 + 8 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 74 (User); Chips: 741; Takeable: 2
[Solve] Chips: 741 = F_15 + F_11 + F_9 + F_6 = 610 + 89 + 34 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 75 (Computer); Chips: 739; Takeable: 4
[Solve] Chips: 739 = F_15 + F_11 + F_9 + F_5 + F_2 = 610 + 89 + 34 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 76 (User); Chips: 738; Takeable: 2
[Solve] Chips: 738 = F_15 + F_11 + F_9 + F_5 = 610 + 89 + 34 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 77 (Computer); Chips: 736; Takeable: 4
[Solve] Chips: 736 = F_15 + F_11 + F_9 + F_4 = 610 + 89 + 34 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 78 (User); Chips: 733; Takeable: 6
[Solve] Chips: 733 = F_15 + F_11 + F_9 = 610 + 89 + 34
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 79 (Computer); Chips: 727; Takeable: 12
[Solve] Chips: 727 = F_15 + F_11 + F_8 + F_5 + F_3 = 610 + 89 + 21 + 5 + 2
[Solve] Optimal: 2; Possible: {2, 7}
[Input] Computer Takes: 2
[State] Turn: 80 (User); Chips: 725; Takeable: 4
[Solve] Chips: 725 = F_15 + F_11 + F_8 + F_5 = 610 + 89 + 21 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 81 (Computer); Chips: 721; Takeable: 8
[Solve] Chips: 721 = F_15 + F_11 + F_8 + F_2 = 610 + 89 + 21 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 82 (User); Chips: 720; Takeable: 2
[Solve] Chips: 720 = F_15 + F_11 + F_8 = 610 + 89 + 21
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 83 (Computer); Chips: 718; Takeable: 4
[Solve] Chips: 718 = F_15 + F_11 + F_7 + F_5 + F_2 = 610 + 89 + 13 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 84 (User); Chips: 717; Takeable: 2
[Solve] Chips: 717 = F_15 + F_11 + F_7 + F_5 = 610 + 89 + 13 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 85 (Computer); Chips: 715; Takeable: 4
[Solve] Chips: 715 = F_15 + F_11 + F_7 + F_4 = 610 + 89 + 13 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 86 (User); Chips: 712; Takeable: 6
[Solve] Chips: 712 = F_15 + F_11 + F_7 = 610 + 89 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 87 (Computer); Chips: 706; Takeable: 12
[Solve] Chips: 706 = F_15 + F_11 + F_5 + F_3 = 610 + 89 + 5 + 2
[Solve] Optimal: 2; Possible: {2, 7}
[Input] Computer Takes: 2
[State] Turn: 88 (User); Chips: 704; Takeable: 4
[Solve] Chips: 704 = F_15 + F_11 + F_5 = 610 + 89 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
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[State] Turn: 89 (Computer); Chips: 700; Takeable: 8
[Solve] Chips: 700 = F_15 + F_11 + F_2 = 610 + 89 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 90 (User); Chips: 699; Takeable: 2
[Solve] Chips: 699 = F_15 + F_11 = 610 + 89
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 91 (Computer); Chips: 697; Takeable: 4
[Solve] Chips: 697 = F_15 + F_10 + F_8 + F_6 + F_4 = 610 + 55 + 21 + 8 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 92 (User); Chips: 694; Takeable: 6
[Solve] Chips: 694 = F_15 + F_10 + F_8 + F_6 = 610 + 55 + 21 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 93 (Computer); Chips: 688; Takeable: 12
[Solve] Chips: 688 = F_15 + F_10 + F_8 + F_3 = 610 + 55 + 21 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 94 (User); Chips: 686; Takeable: 4
[Solve] Chips: 686 = F_15 + F_10 + F_8 = 610 + 55 + 21
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 95 (Computer); Chips: 682; Takeable: 8
[Solve] Chips: 682 = F_15 + F_10 + F_7 + F_4 + F_2 = 610 + 55 + 13 + 3 + 1
[Solve] Optimal: 1; Possible: {1, 4}
[Input] Computer Takes: 1
[State] Turn: 96 (User); Chips: 681; Takeable: 2
[Solve] Chips: 681 = F_15 + F_10 + F_7 + F_4 = 610 + 55 + 13 + 3
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 97 (Computer); Chips: 679; Takeable: 4
[Solve] Chips: 679 = F_15 + F_10 + F_7 + F_2 = 610 + 55 + 13 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 98 (User); Chips: 678; Takeable: 2
[Solve] Chips: 678 = F_15 + F_10 + F_7 = 610 + 55 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 99 (Computer); Chips: 676; Takeable: 4
[Solve] Chips: 676 = F_15 + F_10 + F_6 + F_4 = 610 + 55 + 8 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 100 (User); Chips: 673; Takeable: 6
[Solve] Chips: 673 = F_15 + F_10 + F_6 = 610 + 55 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 101 (Computer); Chips: 667; Takeable: 12
[Solve] Chips: 667 = F_15 + F_10 + F_3 = 610 + 55 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 102 (User); Chips: 665; Takeable: 4
[Solve] Chips: 665 = F_15 + F_10 = 610 + 55
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 103 (Computer); Chips: 661; Takeable: 8
[Solve] Chips: 661 = F_15 + F_9 + F_7 + F_4 + F_2 = 610 + 34 + 13 + 3 + 1
[Solve] Optimal: 1; Possible: {1, 4}
[Input] Computer Takes: 1
[State] Turn: 104 (User); Chips: 660; Takeable: 2
[Solve] Chips: 660 = F_15 + F_9 + F_7 + F_4 = 610 + 34 + 13 + 3
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 105 (Computer); Chips: 658; Takeable: 4
[Solve] Chips: 658 = F_15 + F_9 + F_7 + F_2 = 610 + 34 + 13 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 106 (User); Chips: 657; Takeable: 2
[Solve] Chips: 657 = F_15 + F_9 + F_7 = 610 + 34 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 107 (Computer); Chips: 655; Takeable: 4
[Solve] Chips: 655 = F_15 + F_9 + F_6 + F_4 = 610 + 34 + 8 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 108 (User); Chips: 652; Takeable: 6
[Solve] Chips: 652 = F_15 + F_9 + F_6 = 610 + 34 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 109 (Computer); Chips: 646; Takeable: 12
[Solve] Chips: 646 = F_15 + F_9 + F_3 = 610 + 34 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
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[State] Turn: 110 (User); Chips: 644; Takeable: 4
[Solve] Chips: 644 = F_15 + F_9 = 610 + 34
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 111 (Computer); Chips: 640; Takeable: 8
[Solve] Chips: 640 = F_15 + F_8 + F_6 + F_2 = 610 + 21 + 8 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 112 (User); Chips: 639; Takeable: 2
[Solve] Chips: 639 = F_15 + F_8 + F_6 = 610 + 21 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 113 (Computer); Chips: 637; Takeable: 4
[Solve] Chips: 637 = F_15 + F_8 + F_5 + F_2 = 610 + 21 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 114 (User); Chips: 636; Takeable: 2
[Solve] Chips: 636 = F_15 + F_8 + F_5 = 610 + 21 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 115 (Computer); Chips: 634; Takeable: 4
[Solve] Chips: 634 = F_15 + F_8 + F_4 = 610 + 21 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 116 (User); Chips: 631; Takeable: 6
[Solve] Chips: 631 = F_15 + F_8 = 610 + 21
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 117 (Computer); Chips: 625; Takeable: 12
[Solve] Chips: 625 = F_15 + F_7 + F_3 = 610 + 13 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 118 (User); Chips: 623; Takeable: 4
[Solve] Chips: 623 = F_15 + F_7 = 610 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 119 (Computer); Chips: 619; Takeable: 8
[Solve] Chips: 619 = F_15 + F_6 + F_2 = 610 + 8 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 120 (User); Chips: 618; Takeable: 2
[Solve] Chips: 618 = F_15 + F_6 = 610 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 121 (Computer); Chips: 616; Takeable: 4
[Solve] Chips: 616 = F_15 + F_5 + F_2 = 610 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 122 (User); Chips: 615; Takeable: 2
[Solve] Chips: 615 = F_15 + F_5 = 610 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 123 (Computer); Chips: 613; Takeable: 4
[Solve] Chips: 613 = F_15 + F_4 = 610 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 124 (User); Chips: 610; Takeable: 6
[Solve] Chips: 610 = F_15 = 610
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 125 (Computer); Chips: 604; Takeable: 12
[Solve] Chips: 604 = F_14 + F_12 + F_10 + F_8 + F_5 + F_3 = 377 + 144 + 55 + 21 + 5 + 2
[Solve] Optimal: 2; Possible: {2, 7}
[Input] Computer Takes: 2
[State] Turn: 126 (User); Chips: 602; Takeable: 4
[Solve] Chips: 602 = F_14 + F_12 + F_10 + F_8 + F_5 = 377 + 144 + 55 + 21 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 127 (Computer); Chips: 598; Takeable: 8
[Solve] Chips: 598 = F_14 + F_12 + F_10 + F_8 + F_2 = 377 + 144 + 55 + 21 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 128 (User); Chips: 597; Takeable: 2
[Solve] Chips: 597 = F_14 + F_12 + F_10 + F_8 = 377 + 144 + 55 + 21
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 129 (Computer); Chips: 595; Takeable: 4
[Solve] Chips: 595 = F_14 + F_12 + F_10 + F_7 + F_5 + F_2 = 377 + 144 + 55 + 13 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 130 (User); Chips: 594; Takeable: 2
[Solve] Chips: 594 = F_14 + F_12 + F_10 + F_7 + F_5 = 377 + 144 + 55 + 13 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
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[State] Turn: 131 (Computer); Chips: 592; Takeable: 4
[Solve] Chips: 592 = F_14 + F_12 + F_10 + F_7 + F_4 = 377 + 144 + 55 + 13 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 132 (User); Chips: 589; Takeable: 6
[Solve] Chips: 589 = F_14 + F_12 + F_10 + F_7 = 377 + 144 + 55 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 133 (Computer); Chips: 583; Takeable: 12
[Solve] Chips: 583 = F_14 + F_12 + F_10 + F_5 + F_3 = 377 + 144 + 55 + 5 + 2
[Solve] Optimal: 2; Possible: {2, 7}
[Input] Computer Takes: 2
[State] Turn: 134 (User); Chips: 581; Takeable: 4
[Solve] Chips: 581 = F_14 + F_12 + F_10 + F_5 = 377 + 144 + 55 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 135 (Computer); Chips: 577; Takeable: 8
[Solve] Chips: 577 = F_14 + F_12 + F_10 + F_2 = 377 + 144 + 55 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 136 (User); Chips: 576; Takeable: 2
[Solve] Chips: 576 = F_14 + F_12 + F_10 = 377 + 144 + 55
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 137 (Computer); Chips: 574; Takeable: 4
[Solve] Chips: 574 = F_14 + F_12 + F_9 + F_7 + F_5 + F_2 = 377 + 144 + 34 + 13 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 138 (User); Chips: 573; Takeable: 2
[Solve] Chips: 573 = F_14 + F_12 + F_9 + F_7 + F_5 = 377 + 144 + 34 + 13 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 139 (Computer); Chips: 571; Takeable: 4
[Solve] Chips: 571 = F_14 + F_12 + F_9 + F_7 + F_4 = 377 + 144 + 34 + 13 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 140 (User); Chips: 568; Takeable: 6
[Solve] Chips: 568 = F_14 + F_12 + F_9 + F_7 = 377 + 144 + 34 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 141 (Computer); Chips: 562; Takeable: 12
[Solve] Chips: 562 = F_14 + F_12 + F_9 + F_5 + F_3 = 377 + 144 + 34 + 5 + 2
[Solve] Optimal: 2; Possible: {2, 7}
[Input] Computer Takes: 2
[State] Turn: 142 (User); Chips: 560; Takeable: 4
[Solve] Chips: 560 = F_14 + F_12 + F_9 + F_5 = 377 + 144 + 34 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 143 (Computer); Chips: 556; Takeable: 8
[Solve] Chips: 556 = F_14 + F_12 + F_9 + F_2 = 377 + 144 + 34 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 144 (User); Chips: 555; Takeable: 2
[Solve] Chips: 555 = F_14 + F_12 + F_9 = 377 + 144 + 34
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 145 (Computer); Chips: 553; Takeable: 4
[Solve] Chips: 553 = F_14 + F_12 + F_8 + F_6 + F_4 = 377 + 144 + 21 + 8 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 146 (User); Chips: 550; Takeable: 6
[Solve] Chips: 550 = F_14 + F_12 + F_8 + F_6 = 377 + 144 + 21 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 147 (Computer); Chips: 544; Takeable: 12
[Solve] Chips: 544 = F_14 + F_12 + F_8 + F_3 = 377 + 144 + 21 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 148 (User); Chips: 542; Takeable: 4
[Solve] Chips: 542 = F_14 + F_12 + F_8 = 377 + 144 + 21
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 149 (Computer); Chips: 538; Takeable: 8
[Solve] Chips: 538 = F_14 + F_12 + F_7 + F_4 + F_2 = 377 + 144 + 13 + 3 + 1
[Solve] Optimal: 1; Possible: {1, 4}
[Input] Computer Takes: 1
[State] Turn: 150 (User); Chips: 537; Takeable: 2
[Solve] Chips: 537 = F_14 + F_12 + F_7 + F_4 = 377 + 144 + 13 + 3
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 151 (Computer); Chips: 535; Takeable: 4
[Solve] Chips: 535 = F_14 + F_12 + F_7 + F_2 = 377 + 144 + 13 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
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[State] Turn: 152 (User); Chips: 534; Takeable: 2
[Solve] Chips: 534 = F_14 + F_12 + F_7 = 377 + 144 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 153 (Computer); Chips: 532; Takeable: 4
[Solve] Chips: 532 = F_14 + F_12 + F_6 + F_4 = 377 + 144 + 8 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 154 (User); Chips: 529; Takeable: 6
[Solve] Chips: 529 = F_14 + F_12 + F_6 = 377 + 144 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 155 (Computer); Chips: 523; Takeable: 12
[Solve] Chips: 523 = F_14 + F_12 + F_3 = 377 + 144 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 156 (User); Chips: 521; Takeable: 4
[Solve] Chips: 521 = F_14 + F_12 = 377 + 144
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 157 (Computer); Chips: 517; Takeable: 8
[Solve] Chips: 517 = F_14 + F_11 + F_9 + F_7 + F_4 + F_2 = 377 + 89 + 34 + 13 + 3 + 1
[Solve] Optimal: 1; Possible: {1, 4}
[Input] Computer Takes: 1
[State] Turn: 158 (User); Chips: 516; Takeable: 2
[Solve] Chips: 516 = F_14 + F_11 + F_9 + F_7 + F_4 = 377 + 89 + 34 + 13 + 3
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 159 (Computer); Chips: 514; Takeable: 4
[Solve] Chips: 514 = F_14 + F_11 + F_9 + F_7 + F_2 = 377 + 89 + 34 + 13 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 160 (User); Chips: 513; Takeable: 2
[Solve] Chips: 513 = F_14 + F_11 + F_9 + F_7 = 377 + 89 + 34 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 161 (Computer); Chips: 511; Takeable: 4
[Solve] Chips: 511 = F_14 + F_11 + F_9 + F_6 + F_4 = 377 + 89 + 34 + 8 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 162 (User); Chips: 508; Takeable: 6
[Solve] Chips: 508 = F_14 + F_11 + F_9 + F_6 = 377 + 89 + 34 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 163 (Computer); Chips: 502; Takeable: 12
[Solve] Chips: 502 = F_14 + F_11 + F_9 + F_3 = 377 + 89 + 34 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 164 (User); Chips: 500; Takeable: 4
[Solve] Chips: 500 = F_14 + F_11 + F_9 = 377 + 89 + 34
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 165 (Computer); Chips: 496; Takeable: 8
[Solve] Chips: 496 = F_14 + F_11 + F_8 + F_6 + F_2 = 377 + 89 + 21 + 8 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 166 (User); Chips: 495; Takeable: 2
[Solve] Chips: 495 = F_14 + F_11 + F_8 + F_6 = 377 + 89 + 21 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 167 (Computer); Chips: 493; Takeable: 4
[Solve] Chips: 493 = F_14 + F_11 + F_8 + F_5 + F_2 = 377 + 89 + 21 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 168 (User); Chips: 492; Takeable: 2
[Solve] Chips: 492 = F_14 + F_11 + F_8 + F_5 = 377 + 89 + 21 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 169 (Computer); Chips: 490; Takeable: 4
[Solve] Chips: 490 = F_14 + F_11 + F_8 + F_4 = 377 + 89 + 21 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 170 (User); Chips: 487; Takeable: 6
[Solve] Chips: 487 = F_14 + F_11 + F_8 = 377 + 89 + 21
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 171 (Computer); Chips: 481; Takeable: 12
[Solve] Chips: 481 = F_14 + F_11 + F_7 + F_3 = 377 + 89 + 13 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 172 (User); Chips: 479; Takeable: 4
[Solve] Chips: 479 = F_14 + F_11 + F_7 = 377 + 89 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
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[State] Turn: 173 (Computer); Chips: 475; Takeable: 8
[Solve] Chips: 475 = F_14 + F_11 + F_6 + F_2 = 377 + 89 + 8 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 174 (User); Chips: 474; Takeable: 2
[Solve] Chips: 474 = F_14 + F_11 + F_6 = 377 + 89 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 175 (Computer); Chips: 472; Takeable: 4
[Solve] Chips: 472 = F_14 + F_11 + F_5 + F_2 = 377 + 89 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 176 (User); Chips: 471; Takeable: 2
[Solve] Chips: 471 = F_14 + F_11 + F_5 = 377 + 89 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 177 (Computer); Chips: 469; Takeable: 4
[Solve] Chips: 469 = F_14 + F_11 + F_4 = 377 + 89 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 178 (User); Chips: 466; Takeable: 6
[Solve] Chips: 466 = F_14 + F_11 = 377 + 89
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 179 (Computer); Chips: 460; Takeable: 12
[Solve] Chips: 460 = F_14 + F_10 + F_8 + F_5 + F_3 = 377 + 55 + 21 + 5 + 2
[Solve] Optimal: 2; Possible: {2, 7}
[Input] Computer Takes: 2
[State] Turn: 180 (User); Chips: 458; Takeable: 4
[Solve] Chips: 458 = F_14 + F_10 + F_8 + F_5 = 377 + 55 + 21 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 181 (Computer); Chips: 454; Takeable: 8
[Solve] Chips: 454 = F_14 + F_10 + F_8 + F_2 = 377 + 55 + 21 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 182 (User); Chips: 453; Takeable: 2
[Solve] Chips: 453 = F_14 + F_10 + F_8 = 377 + 55 + 21
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 183 (Computer); Chips: 451; Takeable: 4
[Solve] Chips: 451 = F_14 + F_10 + F_7 + F_5 + F_2 = 377 + 55 + 13 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 184 (User); Chips: 450; Takeable: 2
[Solve] Chips: 450 = F_14 + F_10 + F_7 + F_5 = 377 + 55 + 13 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 185 (Computer); Chips: 448; Takeable: 4
[Solve] Chips: 448 = F_14 + F_10 + F_7 + F_4 = 377 + 55 + 13 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 186 (User); Chips: 445; Takeable: 6
[Solve] Chips: 445 = F_14 + F_10 + F_7 = 377 + 55 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 187 (Computer); Chips: 439; Takeable: 12
[Solve] Chips: 439 = F_14 + F_10 + F_5 + F_3 = 377 + 55 + 5 + 2
[Solve] Optimal: 2; Possible: {2, 7}
[Input] Computer Takes: 2
[State] Turn: 188 (User); Chips: 437; Takeable: 4
[Solve] Chips: 437 = F_14 + F_10 + F_5 = 377 + 55 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 189 (Computer); Chips: 433; Takeable: 8
[Solve] Chips: 433 = F_14 + F_10 + F_2 = 377 + 55 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 190 (User); Chips: 432; Takeable: 2
[Solve] Chips: 432 = F_14 + F_10 = 377 + 55
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 191 (Computer); Chips: 430; Takeable: 4
[Solve] Chips: 430 = F_14 + F_9 + F_7 + F_5 + F_2 = 377 + 34 + 13 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 192 (User); Chips: 429; Takeable: 2
[Solve] Chips: 429 = F_14 + F_9 + F_7 + F_5 = 377 + 34 + 13 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 193 (Computer); Chips: 427; Takeable: 4
[Solve] Chips: 427 = F_14 + F_9 + F_7 + F_4 = 377 + 34 + 13 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
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[State] Turn: 194 (User); Chips: 424; Takeable: 6
[Solve] Chips: 424 = F_14 + F_9 + F_7 = 377 + 34 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 195 (Computer); Chips: 418; Takeable: 12
[Solve] Chips: 418 = F_14 + F_9 + F_5 + F_3 = 377 + 34 + 5 + 2
[Solve] Optimal: 2; Possible: {2, 7}
[Input] Computer Takes: 2
[State] Turn: 196 (User); Chips: 416; Takeable: 4
[Solve] Chips: 416 = F_14 + F_9 + F_5 = 377 + 34 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 197 (Computer); Chips: 412; Takeable: 8
[Solve] Chips: 412 = F_14 + F_9 + F_2 = 377 + 34 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 198 (User); Chips: 411; Takeable: 2
[Solve] Chips: 411 = F_14 + F_9 = 377 + 34
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 199 (Computer); Chips: 409; Takeable: 4
[Solve] Chips: 409 = F_14 + F_8 + F_6 + F_4 = 377 + 21 + 8 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 200 (User); Chips: 406; Takeable: 6
[Solve] Chips: 406 = F_14 + F_8 + F_6 = 377 + 21 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 201 (Computer); Chips: 400; Takeable: 12
[Solve] Chips: 400 = F_14 + F_8 + F_3 = 377 + 21 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 202 (User); Chips: 398; Takeable: 4
[Solve] Chips: 398 = F_14 + F_8 = 377 + 21
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 203 (Computer); Chips: 394; Takeable: 8
[Solve] Chips: 394 = F_14 + F_7 + F_4 + F_2 = 377 + 13 + 3 + 1
[Solve] Optimal: 1; Possible: {1, 4}
[Input] Computer Takes: 1
[State] Turn: 204 (User); Chips: 393; Takeable: 2
[Solve] Chips: 393 = F_14 + F_7 + F_4 = 377 + 13 + 3
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 205 (Computer); Chips: 391; Takeable: 4
[Solve] Chips: 391 = F_14 + F_7 + F_2 = 377 + 13 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 206 (User); Chips: 390; Takeable: 2
[Solve] Chips: 390 = F_14 + F_7 = 377 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 207 (Computer); Chips: 388; Takeable: 4
[Solve] Chips: 388 = F_14 + F_6 + F_4 = 377 + 8 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 208 (User); Chips: 385; Takeable: 6
[Solve] Chips: 385 = F_14 + F_6 = 377 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 209 (Computer); Chips: 379; Takeable: 12
[Solve] Chips: 379 = F_14 + F_3 = 377 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 210 (User); Chips: 377; Takeable: 4
[Solve] Chips: 377 = F_14 = 377
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 211 (Computer); Chips: 373; Takeable: 8
[Solve] Chips: 373 = F_13 + F_11 + F_9 + F_7 + F_4 + F_2 = 233 + 89 + 34 + 13 + 3 + 1
[Solve] Optimal: 1; Possible: {1, 4}
[Input] Computer Takes: 1
[State] Turn: 212 (User); Chips: 372; Takeable: 2
[Solve] Chips: 372 = F_13 + F_11 + F_9 + F_7 + F_4 = 233 + 89 + 34 + 13 + 3
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 213 (Computer); Chips: 370; Takeable: 4
[Solve] Chips: 370 = F_13 + F_11 + F_9 + F_7 + F_2 = 233 + 89 + 34 + 13 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 214 (User); Chips: 369; Takeable: 2
[Solve] Chips: 369 = F_13 + F_11 + F_9 + F_7 = 233 + 89 + 34 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
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[State] Turn: 215 (Computer); Chips: 367; Takeable: 4
[Solve] Chips: 367 = F_13 + F_11 + F_9 + F_6 + F_4 = 233 + 89 + 34 + 8 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 216 (User); Chips: 364; Takeable: 6
[Solve] Chips: 364 = F_13 + F_11 + F_9 + F_6 = 233 + 89 + 34 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 217 (Computer); Chips: 358; Takeable: 12
[Solve] Chips: 358 = F_13 + F_11 + F_9 + F_3 = 233 + 89 + 34 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 218 (User); Chips: 356; Takeable: 4
[Solve] Chips: 356 = F_13 + F_11 + F_9 = 233 + 89 + 34
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 219 (Computer); Chips: 352; Takeable: 8
[Solve] Chips: 352 = F_13 + F_11 + F_8 + F_6 + F_2 = 233 + 89 + 21 + 8 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 220 (User); Chips: 351; Takeable: 2
[Solve] Chips: 351 = F_13 + F_11 + F_8 + F_6 = 233 + 89 + 21 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 221 (Computer); Chips: 349; Takeable: 4
[Solve] Chips: 349 = F_13 + F_11 + F_8 + F_5 + F_2 = 233 + 89 + 21 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 222 (User); Chips: 348; Takeable: 2
[Solve] Chips: 348 = F_13 + F_11 + F_8 + F_5 = 233 + 89 + 21 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 223 (Computer); Chips: 346; Takeable: 4
[Solve] Chips: 346 = F_13 + F_11 + F_8 + F_4 = 233 + 89 + 21 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 224 (User); Chips: 343; Takeable: 6
[Solve] Chips: 343 = F_13 + F_11 + F_8 = 233 + 89 + 21
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 225 (Computer); Chips: 337; Takeable: 12
[Solve] Chips: 337 = F_13 + F_11 + F_7 + F_3 = 233 + 89 + 13 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 226 (User); Chips: 335; Takeable: 4
[Solve] Chips: 335 = F_13 + F_11 + F_7 = 233 + 89 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 227 (Computer); Chips: 331; Takeable: 8
[Solve] Chips: 331 = F_13 + F_11 + F_6 + F_2 = 233 + 89 + 8 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 228 (User); Chips: 330; Takeable: 2
[Solve] Chips: 330 = F_13 + F_11 + F_6 = 233 + 89 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 229 (Computer); Chips: 328; Takeable: 4
[Solve] Chips: 328 = F_13 + F_11 + F_5 + F_2 = 233 + 89 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 230 (User); Chips: 327; Takeable: 2
[Solve] Chips: 327 = F_13 + F_11 + F_5 = 233 + 89 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 231 (Computer); Chips: 325; Takeable: 4
[Solve] Chips: 325 = F_13 + F_11 + F_4 = 233 + 89 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 232 (User); Chips: 322; Takeable: 6
[Solve] Chips: 322 = F_13 + F_11 = 233 + 89
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 233 (Computer); Chips: 316; Takeable: 12
[Solve] Chips: 316 = F_13 + F_10 + F_8 + F_5 + F_3 = 233 + 55 + 21 + 5 + 2
[Solve] Optimal: 2; Possible: {2, 7}
[Input] Computer Takes: 2
[State] Turn: 234 (User); Chips: 314; Takeable: 4
[Solve] Chips: 314 = F_13 + F_10 + F_8 + F_5 = 233 + 55 + 21 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 235 (Computer); Chips: 310; Takeable: 8
[Solve] Chips: 310 = F_13 + F_10 + F_8 + F_2 = 233 + 55 + 21 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
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[State] Turn: 236 (User); Chips: 309; Takeable: 2
[Solve] Chips: 309 = F_13 + F_10 + F_8 = 233 + 55 + 21
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 237 (Computer); Chips: 307; Takeable: 4
[Solve] Chips: 307 = F_13 + F_10 + F_7 + F_5 + F_2 = 233 + 55 + 13 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 238 (User); Chips: 306; Takeable: 2
[Solve] Chips: 306 = F_13 + F_10 + F_7 + F_5 = 233 + 55 + 13 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 239 (Computer); Chips: 304; Takeable: 4
[Solve] Chips: 304 = F_13 + F_10 + F_7 + F_4 = 233 + 55 + 13 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 240 (User); Chips: 301; Takeable: 6
[Solve] Chips: 301 = F_13 + F_10 + F_7 = 233 + 55 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 241 (Computer); Chips: 295; Takeable: 12
[Solve] Chips: 295 = F_13 + F_10 + F_5 + F_3 = 233 + 55 + 5 + 2
[Solve] Optimal: 2; Possible: {2, 7}
[Input] Computer Takes: 2
[State] Turn: 242 (User); Chips: 293; Takeable: 4
[Solve] Chips: 293 = F_13 + F_10 + F_5 = 233 + 55 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 243 (Computer); Chips: 289; Takeable: 8
[Solve] Chips: 289 = F_13 + F_10 + F_2 = 233 + 55 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 244 (User); Chips: 288; Takeable: 2
[Solve] Chips: 288 = F_13 + F_10 = 233 + 55
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 245 (Computer); Chips: 286; Takeable: 4
[Solve] Chips: 286 = F_13 + F_9 + F_7 + F_5 + F_2 = 233 + 34 + 13 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 246 (User); Chips: 285; Takeable: 2
[Solve] Chips: 285 = F_13 + F_9 + F_7 + F_5 = 233 + 34 + 13 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 247 (Computer); Chips: 283; Takeable: 4
[Solve] Chips: 283 = F_13 + F_9 + F_7 + F_4 = 233 + 34 + 13 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 248 (User); Chips: 280; Takeable: 6
[Solve] Chips: 280 = F_13 + F_9 + F_7 = 233 + 34 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 249 (Computer); Chips: 274; Takeable: 12
[Solve] Chips: 274 = F_13 + F_9 + F_5 + F_3 = 233 + 34 + 5 + 2
[Solve] Optimal: 2; Possible: {2, 7}
[Input] Computer Takes: 2
[State] Turn: 250 (User); Chips: 272; Takeable: 4
[Solve] Chips: 272 = F_13 + F_9 + F_5 = 233 + 34 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 251 (Computer); Chips: 268; Takeable: 8
[Solve] Chips: 268 = F_13 + F_9 + F_2 = 233 + 34 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 252 (User); Chips: 267; Takeable: 2
[Solve] Chips: 267 = F_13 + F_9 = 233 + 34
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 253 (Computer); Chips: 265; Takeable: 4
[Solve] Chips: 265 = F_13 + F_8 + F_6 + F_4 = 233 + 21 + 8 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 254 (User); Chips: 262; Takeable: 6
[Solve] Chips: 262 = F_13 + F_8 + F_6 = 233 + 21 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 255 (Computer); Chips: 256; Takeable: 12
[Solve] Chips: 256 = F_13 + F_8 + F_3 = 233 + 21 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 256 (User); Chips: 254; Takeable: 4
[Solve] Chips: 254 = F_13 + F_8 = 233 + 21
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
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[State] Turn: 257 (Computer); Chips: 250; Takeable: 8
[Solve] Chips: 250 = F_13 + F_7 + F_4 + F_2 = 233 + 13 + 3 + 1
[Solve] Optimal: 1; Possible: {1, 4}
[Input] Computer Takes: 1
[State] Turn: 258 (User); Chips: 249; Takeable: 2
[Solve] Chips: 249 = F_13 + F_7 + F_4 = 233 + 13 + 3
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 259 (Computer); Chips: 247; Takeable: 4
[Solve] Chips: 247 = F_13 + F_7 + F_2 = 233 + 13 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 260 (User); Chips: 246; Takeable: 2
[Solve] Chips: 246 = F_13 + F_7 = 233 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 261 (Computer); Chips: 244; Takeable: 4
[Solve] Chips: 244 = F_13 + F_6 + F_4 = 233 + 8 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 262 (User); Chips: 241; Takeable: 6
[Solve] Chips: 241 = F_13 + F_6 = 233 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 263 (Computer); Chips: 235; Takeable: 12
[Solve] Chips: 235 = F_13 + F_3 = 233 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 264 (User); Chips: 233; Takeable: 4
[Solve] Chips: 233 = F_13 = 233
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 265 (Computer); Chips: 229; Takeable: 8
[Solve] Chips: 229 = F_12 + F_10 + F_8 + F_6 + F_2 = 144 + 55 + 21 + 8 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 266 (User); Chips: 228; Takeable: 2
[Solve] Chips: 228 = F_12 + F_10 + F_8 + F_6 = 144 + 55 + 21 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 267 (Computer); Chips: 226; Takeable: 4
[Solve] Chips: 226 = F_12 + F_10 + F_8 + F_5 + F_2 = 144 + 55 + 21 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 268 (User); Chips: 225; Takeable: 2
[Solve] Chips: 225 = F_12 + F_10 + F_8 + F_5 = 144 + 55 + 21 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 269 (Computer); Chips: 223; Takeable: 4
[Solve] Chips: 223 = F_12 + F_10 + F_8 + F_4 = 144 + 55 + 21 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 270 (User); Chips: 220; Takeable: 6
[Solve] Chips: 220 = F_12 + F_10 + F_8 = 144 + 55 + 21
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 271 (Computer); Chips: 214; Takeable: 12
[Solve] Chips: 214 = F_12 + F_10 + F_7 + F_3 = 144 + 55 + 13 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 272 (User); Chips: 212; Takeable: 4
[Solve] Chips: 212 = F_12 + F_10 + F_7 = 144 + 55 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 273 (Computer); Chips: 208; Takeable: 8
[Solve] Chips: 208 = F_12 + F_10 + F_6 + F_2 = 144 + 55 + 8 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 274 (User); Chips: 207; Takeable: 2
[Solve] Chips: 207 = F_12 + F_10 + F_6 = 144 + 55 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 275 (Computer); Chips: 205; Takeable: 4
[Solve] Chips: 205 = F_12 + F_10 + F_5 + F_2 = 144 + 55 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 276 (User); Chips: 204; Takeable: 2
[Solve] Chips: 204 = F_12 + F_10 + F_5 = 144 + 55 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 277 (Computer); Chips: 202; Takeable: 4
[Solve] Chips: 202 = F_12 + F_10 + F_4 = 144 + 55 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
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[State] Turn: 278 (User); Chips: 199; Takeable: 6
[Solve] Chips: 199 = F_12 + F_10 = 144 + 55
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 279 (Computer); Chips: 193; Takeable: 12
[Solve] Chips: 193 = F_12 + F_9 + F_7 + F_3 = 144 + 34 + 13 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 280 (User); Chips: 191; Takeable: 4
[Solve] Chips: 191 = F_12 + F_9 + F_7 = 144 + 34 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 281 (Computer); Chips: 187; Takeable: 8
[Solve] Chips: 187 = F_12 + F_9 + F_6 + F_2 = 144 + 34 + 8 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 282 (User); Chips: 186; Takeable: 2
[Solve] Chips: 186 = F_12 + F_9 + F_6 = 144 + 34 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 283 (Computer); Chips: 184; Takeable: 4
[Solve] Chips: 184 = F_12 + F_9 + F_5 + F_2 = 144 + 34 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 284 (User); Chips: 183; Takeable: 2
[Solve] Chips: 183 = F_12 + F_9 + F_5 = 144 + 34 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 285 (Computer); Chips: 181; Takeable: 4
[Solve] Chips: 181 = F_12 + F_9 + F_4 = 144 + 34 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 286 (User); Chips: 178; Takeable: 6
[Solve] Chips: 178 = F_12 + F_9 = 144 + 34
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 287 (Computer); Chips: 172; Takeable: 12
[Solve] Chips: 172 = F_12 + F_8 + F_5 + F_3 = 144 + 21 + 5 + 2
[Solve] Optimal: 2; Possible: {2, 7}
[Input] Computer Takes: 2
[State] Turn: 288 (User); Chips: 170; Takeable: 4
[Solve] Chips: 170 = F_12 + F_8 + F_5 = 144 + 21 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 289 (Computer); Chips: 166; Takeable: 8
[Solve] Chips: 166 = F_12 + F_8 + F_2 = 144 + 21 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 290 (User); Chips: 165; Takeable: 2
[Solve] Chips: 165 = F_12 + F_8 = 144 + 21
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 291 (Computer); Chips: 163; Takeable: 4
[Solve] Chips: 163 = F_12 + F_7 + F_5 + F_2 = 144 + 13 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 292 (User); Chips: 162; Takeable: 2
[Solve] Chips: 162 = F_12 + F_7 + F_5 = 144 + 13 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 293 (Computer); Chips: 160; Takeable: 4
[Solve] Chips: 160 = F_12 + F_7 + F_4 = 144 + 13 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 294 (User); Chips: 157; Takeable: 6
[Solve] Chips: 157 = F_12 + F_7 = 144 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 295 (Computer); Chips: 151; Takeable: 12
[Solve] Chips: 151 = F_12 + F_5 + F_3 = 144 + 5 + 2
[Solve] Optimal: 2; Possible: {2, 7}
[Input] Computer Takes: 2
[State] Turn: 296 (User); Chips: 149; Takeable: 4
[Solve] Chips: 149 = F_12 + F_5 = 144 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 297 (Computer); Chips: 145; Takeable: 8
[Solve] Chips: 145 = F_12 + F_2 = 144 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 298 (User); Chips: 144; Takeable: 2
[Solve] Chips: 144 = F_12 = 144
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
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[State] Turn: 299 (Computer); Chips: 142; Takeable: 4
[Solve] Chips: 142 = F_11 + F_9 + F_7 + F_5 + F_2 = 89 + 34 + 13 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 300 (User); Chips: 141; Takeable: 2
[Solve] Chips: 141 = F_11 + F_9 + F_7 + F_5 = 89 + 34 + 13 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 301 (Computer); Chips: 139; Takeable: 4
[Solve] Chips: 139 = F_11 + F_9 + F_7 + F_4 = 89 + 34 + 13 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 302 (User); Chips: 136; Takeable: 6
[Solve] Chips: 136 = F_11 + F_9 + F_7 = 89 + 34 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 303 (Computer); Chips: 130; Takeable: 12
[Solve] Chips: 130 = F_11 + F_9 + F_5 + F_3 = 89 + 34 + 5 + 2
[Solve] Optimal: 2; Possible: {2, 7}
[Input] Computer Takes: 2
[State] Turn: 304 (User); Chips: 128; Takeable: 4
[Solve] Chips: 128 = F_11 + F_9 + F_5 = 89 + 34 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 305 (Computer); Chips: 124; Takeable: 8
[Solve] Chips: 124 = F_11 + F_9 + F_2 = 89 + 34 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 306 (User); Chips: 123; Takeable: 2
[Solve] Chips: 123 = F_11 + F_9 = 89 + 34
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 307 (Computer); Chips: 121; Takeable: 4
[Solve] Chips: 121 = F_11 + F_8 + F_6 + F_4 = 89 + 21 + 8 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 308 (User); Chips: 118; Takeable: 6
[Solve] Chips: 118 = F_11 + F_8 + F_6 = 89 + 21 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 309 (Computer); Chips: 112; Takeable: 12
[Solve] Chips: 112 = F_11 + F_8 + F_3 = 89 + 21 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 310 (User); Chips: 110; Takeable: 4
[Solve] Chips: 110 = F_11 + F_8 = 89 + 21
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 311 (Computer); Chips: 106; Takeable: 8
[Solve] Chips: 106 = F_11 + F_7 + F_4 + F_2 = 89 + 13 + 3 + 1
[Solve] Optimal: 1; Possible: {1, 4}
[Input] Computer Takes: 1
[State] Turn: 312 (User); Chips: 105; Takeable: 2
[Solve] Chips: 105 = F_11 + F_7 + F_4 = 89 + 13 + 3
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 313 (Computer); Chips: 103; Takeable: 4
[Solve] Chips: 103 = F_11 + F_7 + F_2 = 89 + 13 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 314 (User); Chips: 102; Takeable: 2
[Solve] Chips: 102 = F_11 + F_7 = 89 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 315 (Computer); Chips: 100; Takeable: 4
[Solve] Chips: 100 = F_11 + F_6 + F_4 = 89 + 8 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 316 (User); Chips: 97; Takeable: 6
[Solve] Chips: 97 = F_11 + F_6 = 89 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 317 (Computer); Chips: 91; Takeable: 12
[Solve] Chips: 91 = F_11 + F_3 = 89 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 318 (User); Chips: 89; Takeable: 4
[Solve] Chips: 89 = F_11 = 89
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 319 (Computer); Chips: 85; Takeable: 8
[Solve] Chips: 85 = F_10 + F_8 + F_6 + F_2 = 55 + 21 + 8 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
```

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[State] Turn: 320 (User); Chips: 84; Takeable: 2
[Solve] Chips: 84 = F_10 + F_8 + F_6 = 55 + 21 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 321 (Computer); Chips: 82; Takeable: 4
[Solve] Chips: 82 = F_10 + F_8 + F_5 + F_2 = 55 + 21 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 322 (User); Chips: 81; Takeable: 2
[Solve] Chips: 81 = F_10 + F_8 + F_5 = 55 + 21 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 323 (Computer); Chips: 79; Takeable: 4
[Solve] Chips: 79 = F_10 + F_8 + F_4 = 55 + 21 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 324 (User); Chips: 76; Takeable: 6
[Solve] Chips: 76 = F_10 + F_8 = 55 + 21
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 325 (Computer); Chips: 70; Takeable: 12
[Solve] Chips: 70 = F_10 + F_7 + F_3 = 55 + 13 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 326 (User); Chips: 68; Takeable: 4
[Solve] Chips: 68 = F_10 + F_7 = 55 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 327 (Computer); Chips: 64; Takeable: 8
[Solve] Chips: 64 = F_10 + F_6 + F_2 = 55 + 8 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 328 (User); Chips: 63; Takeable: 2
[Solve] Chips: 63 = F_10 + F_6 = 55 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 329 (Computer); Chips: 61; Takeable: 4
[Solve] Chips: 61 = F_10 + F_5 + F_2 = 55 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 330 (User); Chips: 60; Takeable: 2
[Solve] Chips: 60 = F_10 + F_5 = 55 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 331 (Computer); Chips: 58; Takeable: 4
[Solve] Chips: 58 = F_10 + F_4 = 55 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 332 (User); Chips: 55; Takeable: 6
[Solve] Chips: 55 = F_10 = 55
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 333 (Computer); Chips: 49; Takeable: 12
[Solve] Chips: 49 = F_9 + F_7 + F_3 = 34 + 13 + 2
[Solve] Optimal: 2; Possible: {2}
[Input] Computer Takes: 2
[State] Turn: 334 (User); Chips: 47; Takeable: 4
[Solve] Chips: 47 = F_9 + F_7 = 34 + 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 335 (Computer); Chips: 43; Takeable: 8
[Solve] Chips: 43 = F_9 + F_6 + F_2 = 34 + 8 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 336 (User); Chips: 42; Takeable: 2
[Solve] Chips: 42 = F_9 + F_6 = 34 + 8
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 337 (Computer); Chips: 40; Takeable: 4
[Solve] Chips: 40 = F_9 + F_5 + F_2 = 34 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 338 (User); Chips: 39; Takeable: 2
[Solve] Chips: 39 = F_9 + F_5 = 34 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 339 (Computer); Chips: 37; Takeable: 4
[Solve] Chips: 37 = F_9 + F_4 = 34 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 340 (User); Chips: 34; Takeable: 6
[Solve] Chips: 34 = F_9 = 34
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
```

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[State] Turn: 341 (Computer); Chips: 28; Takeable: 12
[Solve] Chips: 28 = F_8 + F_5 + F_3 = 21 + 5 + 2
[Solve] Optimal: 2; Possible: {2, 7}
[Input] Computer Takes: 2
[State] Turn: 342 (User); Chips: 26; Takeable: 4
[Solve] Chips: 26 = F_8 + F_5 = 21 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 343 (Computer); Chips: 22; Takeable: 8
[Solve] Chips: 22 = F_8 + F_2 = 21 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 344 (User); Chips: 21; Takeable: 2
[Solve] Chips: 21 = F_8 = 21
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 345 (Computer); Chips: 19; Takeable: 4
[Solve] Chips: 19 = F_7 + F_5 + F_2 = 13 + 5 + 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
[State] Turn: 346 (User); Chips: 18; Takeable: 2
[Solve] Chips: 18 = F_7 + F_5 = 13 + 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 2
[State] Turn: 347 (Computer); Chips: 16; Takeable: 4
[Solve] Chips: 16 = F_7 + F_4 = 13 + 3
[Solve] Optimal: 3; Possible: {3}
[Input] Computer Takes: 3
[State] Turn: 348 (User); Chips: 13; Takeable: 6
[Solve] Chips: 13 = F_7 = 13
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 6
[State] Turn: 349 (Computer); Chips: 7; Takeable: 12
[Solve] Chips: 7 = F_5 + F_3 = 5 + 2
[Solve] Optimal: 2; Possible: {2, 7}
[Input] Computer Takes: 2
[State] Turn: 350 (User); Chips: 5; Takeable: 4
[Solve] Chips: 5 = F_5 = 5
[Solve] Optimal: 1; Possible: {}
[Input] User Takes: 4
[State] Turn: 351 (Computer); Chips: 1; Takeable: 8
[Solve] Chips: 1 = F_2 = 1
[Solve] Optimal: 1; Possible: {1}
[Input] Computer Takes: 1
------
 Turns: 351
Winner: Computer
```

**39.** [*M24*] Find a closed form expression for $a_n$, given that $a_0 = 0$, $a_1 = 1$, and $a_{n+2} = a_{n+1} + 6a_n$ for $n \geq 0$.

We may use the method of generating functions to find a closed form expression for $a_n$. Let

$$G(z) = \sum_{k \geq 0} a_k z^k.$$

Then,

$$\begin{aligned}(1 - z - 6z^2)G(z) &= a_0 z^0 + (a_1 - a_0)z^1 + \sum_{k \geq 2}(a_k - a_{k-1} - 6a_{k-2})z^k \\ &= a_0 z^0 + (a_1 - a_0)z^1 \\ &= 0 + (1 - 0)z \\ &= z,\end{aligned}$$

or equivalently, using partial fractions,

$$\begin{aligned}
G(z) &= \frac{z}{1-z-6z^2} \\
&= \frac{z}{-(3z-1)(2z+1)} \\
&= \frac{1}{5}\frac{-1}{-(3z-1)} + \frac{-1}{5}\frac{1}{2z+1} \\
&= \frac{1}{5}\left(\frac{1}{1-3z} - \frac{1}{1-(-2)z}\right) \\
&= \frac{1}{5}\left(\sum_{k\geq 0} 3^k z^k - \sum_{k\geq 0}(-2)^k z^k\right) \\
&= \sum_{k\geq 0} \frac{1}{5}\left(3^k - (-2)^k\right) z^k.
\end{aligned}$$

That is,

$$a_n = (3^n - (-2)^n)/5.$$

**40.** [*M25*] Solve the recurrence

$$f(1) = 0; \qquad f(n) = \min_{0<k<n} \max(1 + f(k), 2 + f(n-k)), \quad \text{for } n > 1.$$

We have that

$$f(n) = m$$

for $0 \leq F_m < n \leq F_{m+1}$, as shown below.

In the case that $m = 0$,

$$f(1) = 0,$$

and

$$F_0 = 0 < 1 \leq 1 = F_1 = F_{0+1}.$$

In the case that $m = 1$,

$$\begin{aligned}
f(2) &= \min_{0<k<2} \max(1 + f(k), 2 + f(2-k)) \\
&= \max(1 + f(1), 2 + f(2-1)) \\
&= \max(1, 2) \\
&= 2,
\end{aligned}$$

and

$$F_2 = 1 < 2 \leq 2 = F_3 = F_{2+1}.$$

Then, assuming

$$f(n) = m$$

for $F_m < n \leq F_{m+1}$, we must show that

$$f(n') = m + 1$$

for $F_{m+1} < n' \leq F_{m+2}$. Note that since $f(n') = \min_{0<k<n'} \max(1 + f(k), 2 + f(n' - k))$, we must have that $f(n') \leq \max(1 + f(k), 2 + f(n' - k))$ for $0 < k < n'$, including for $k = F_{m+1}$, since

$F_{m+1} > 0$ and $F_{m+1} < n'$ by hypothesis. That is, since $f(F_{m+1}) = m$ for $F_m < F_{m+1} \le F_{m+1}$, and since $f(n' - F_{m+1}) \le m - 1$ for $0 < n' - F_{m+1} \le F_m$,

$$\begin{aligned} f(n') &\le \max(1 + f(F_{m+1}), 2 + f(n' - F_{m+1})) \\ &= \max(1 + m, 2 + (m - 1)) \\ &= \max(m + 1, m + 1) \\ &= m + 1. \end{aligned}$$

Then, to see why $f(n') \not< m + 1$, assume it is. Then there must exist some integer $k < n'$ such that $f(k) < m$, so that $k \le F_m$; and such that $f(n' - k) < m - 1$, so that $n' - k \le F_{m-1}$. Then $k + n' - k = n' < F_m + F_{m-1} = F_{m+1}$. But $F_{m+1} < n'$ by the inductive hypothesis. That is, the assumption that $f(n') < m + 1$ leads to a contradiction, allowing us to instead conclude that

$$f(n') = m + 1,$$

as we needed to show.

---

[section 6.2.1]

► **41.** [*M25*] (Yuri Matiyasevich, 1990.) Let $f(x) = \lfloor x + \phi^{-1} \rfloor$. Prove that if $n = F_{k_1} + \cdots + F_{k_r}$ is the representation of $n$ in the Fibonacci number system of exercise 34, then $F_{k_1+1} + \cdots + F_{k_r+1} = f(\phi n)$. Find a similar formula for $F_{k_1-1} + \cdots + F_{k_r-1}$.

We may prove the equality.

**Proposition.** *$\sum_{1 \le j \le r} F_{k_j+1} = \lfloor \phi^{-1} + \phi \sum_{1 \le j \le r} F_{k_j} \rfloor$ if $k_j > k_{j+1} + 1$ for $1 \le j < r$ and $k_r > 1$.*

*Proof.* Let

$$n = \sum_{1 \le j \le r} F_{k_j}$$

be the unique Fibonacci representation of $n$, $k_j > k_{j+1} + 1$ for $1 \le j < r$ and $k_r > 1$. We must show that

$$\sum_{1 \le j \le r} F_{k_j+1} = \lfloor \phi^{-1} + \phi \sum_{1 \le j \le r} F_{k_j} \rfloor = \lfloor \phi n + \phi^{-1} \rfloor.$$

From exercise 11,

$$\begin{aligned} &\hat\phi^{k_j+1} = F_{k_j+1}\hat\phi + F_{k_j} \\ \iff \quad & F_{k_j+1}\hat\phi = \hat\phi^{k_j+1} - F_{k_j} \\ \iff \quad & F_{k_j+1} = \hat\phi^{k_j} - \hat\phi^{-1} F_{k_j} \\ \iff \quad & F_{k_j+1} = \hat\phi^{k_j} + \phi F_{k_j}. \end{aligned}$$

Then

$$\begin{aligned}\sum_{1\le j\le r} F_{k_j+1} &= \sum_{1\le j\le r}\left(\hat\phi^{k_j} + \phi F_{k_j}\right)\\ &= \sum_{1\le j\le r}\hat\phi^{k_j} + \sum_{1\le j\le r}\phi F_{k_j}\\ &= \sum_{1\le j\le r}\hat\phi^{k_j} + \phi\sum_{1\le j\le r}F_{k_j}\\ &= \sum_{1\le j\le r}\hat\phi^{k_j} + \phi n\\ &= \phi n + \sum_{1\le j\le r}\hat\phi^{k_j}.\end{aligned}$$

But $\hat\phi < 0$, so if $k_j > 1$ is even, $\hat\phi^{k_j} > 0$; or if $k_j > 1$ is odd, $\hat\phi^{k_j} < 0$. This determines the upper and lower bounds of the sum of $\hat\phi^{k_j}$ as

$$\sum_{\substack{3\le k\\ k\text{ odd}}}\hat\phi^k < \sum_{1\le j\le r}\hat\phi^{k_j} \le \sum_{\substack{2\le k\\ k\text{ even}}}\hat\phi^k,$$

since $\sum_{\substack{3\le k\\ k\text{ odd}}}\hat\phi^k$ is strictly less than $\sum_{1\le j\le r}\hat\phi^{k_j}$ given an infinite number of terms. But

$$\begin{aligned}\sum_{\substack{3\le k\\ k\text{ odd}}}\hat\phi^k &= \sum_{\substack{3\le k\\ k\text{ odd}}}\left(\hat\phi^{k+1} - \hat\phi^{k-1}\right)\\ &= -\hat\phi^{3-1}\\ &= -\hat\phi^2\\ &= -\left(\hat\phi^1 + \hat\phi^0\right)\\ &= -\hat\phi^1 - 1\\ &= \phi^{-1} - 1\end{aligned}$$

and

$$\begin{aligned}\sum_{\substack{2\le k\\ k\text{ even}}}\hat\phi^k &= \sum_{\substack{2\le k\\ k\text{ even}}}\left(\hat\phi^{k+1} - \hat\phi^{k-1}\right)\\ &= -\hat\phi^{2-1}\\ &= -\hat\phi^1\\ &= \phi^{-1}.\end{aligned}$$

so that

$$\phi^{-1} - 1 < \sum_{1\le j\le r}\hat\phi^{k_j} \le \phi^{-1}.$$

That is,

$$\begin{aligned}
&\phi^{-1} - 1 < \sum_{1 \le j \le r} \hat{\phi}^{k_j} \le \phi^{-1} \\
&\quad\iff \quad \phi n + \phi^{-1} - 1 < \phi n + \sum_{1 \le j \le r} \hat{\phi}^{k_j} \le \phi n + \phi^{-1} \\
&\quad\iff \quad \phi n + \phi^{-1} - 1 < \sum_{1 \le j \le r} F_{k_j+1} \le \phi n + \phi^{-1} \\
&\quad\iff \quad \sum_{1 \le j \le r} F_{k_j+1} = \lfloor \phi n + \phi^{-1} \rfloor
\end{aligned}$$

as we needed to show. □

The formula for $\sum_{1 \le j \le r} F_{k_j - 1}$ is similar,

$$\sum_{1 \le j \le r} F_{k_j-1} = \lfloor \phi^{-1} + \phi^{-1} \sum_{1 \le j \le r} F_{k_j} \rfloor,$$

as shown below.

**Proposition.** $\sum_{1 \le j \le r} F_{k_j-1} = \lfloor \phi^{-1} + \phi^{-1} \sum_{1 \le j \le r} F_{k_j} \rfloor$ *if* $k_j > k_{j+1} + 1$ *for* $1 \le j < r$ *and* $k_r > 1$.

*Proof.* Let

$$n = \sum_{1 \le j \le r} F_{k_j}$$

be the unique Fibonacci representation of $n$, $k_j > k_{j+1} + 1$ for $1 \le j < r$ and $k_r > 1$. We must show that

$$\sum_{1 \le j \le r} F_{k_j-1} = \lfloor \phi^{-1} + \phi^{-1} \sum_{1 \le j \le r} F_{k_j} \rfloor = \lfloor \phi^{-1} n + \phi^{-1} \rfloor.$$

From exercise 11,

$$\begin{aligned}
&\hat{\phi}^{k_j} = F_{k_j} \hat{\phi} + F_{k_j-1} \\
&\quad\iff \quad F_{k_j-1} = \hat{\phi}^{k_j} - F_{k_j} \hat{\phi} \\
&\quad\iff \quad F_{k_j-1} = \hat{\phi}^{k_j} - \hat{\phi} F_{k_j} \\
&\quad\iff \quad F_{k_j-1} = \hat{\phi}^{k_j} + \phi^{-1} F_{k_j}.
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{1 \le j \le r} F_{k_j-1} &= \sum_{1 \le j \le r} \left( \hat{\phi}^{k_j} + \phi^{-1} F_{k_j} \right) \\
&= \sum_{1 \le j \le r} \hat{\phi}^{k_j} + \sum_{1 \le j \le r} \phi^{-1} F_{k_j} \\
&= \sum_{1 \le j \le r} \hat{\phi}^{k_j} + \phi^{-1} \sum_{1 \le j \le r} F_{k_j} \\
&= \sum_{1 \le j \le r} \hat{\phi}^{k_j} + \phi^{-1} n \\
&= \phi^{-1} n + \sum_{1 \le j \le r} \hat{\phi}^{k_j}.
\end{aligned}$$

But $\hat\phi < 0$, so if $k_j > 1$ is even, $\hat\phi^{k_j} > 0$; or if $k_j > 1$ is odd, $\hat\phi^{k_j} < 0$. This determines the upper and lower bounds of the sum of $\hat\phi^{k_j}$ as

$$\sum_{\substack{3\le k\\ k\text{ odd}}} \hat\phi^k < \sum_{1\le j\le r} \hat\phi^{k_j} \le \sum_{\substack{2\le k\\ k\text{ even}}} \hat\phi^k,$$

since $\sum_{\substack{3\le k\\ k\text{ odd}}} \hat\phi^k$ is strictly less than $\sum_{1\le j\le r} \hat\phi^{k_j}$ given an infinite number of terms. But

$$\begin{aligned}\sum_{\substack{3\le k\\ k\text{ odd}}} \hat\phi^k &= \sum_{\substack{3\le k\\ k\text{ odd}}} \left(\hat\phi^{k+1} - \hat\phi^{k-1}\right)\\ &= -\hat\phi^{3-1}\\ &= -\hat\phi^2\\ &= -\left(\hat\phi^1 + \hat\phi^0\right)\\ &= -\hat\phi^1 - 1\\ &= \phi^{-1} - 1\end{aligned}$$

and

$$\begin{aligned}\sum_{\substack{2\le k\\ k\text{ even}}} \hat\phi^k &= \sum_{\substack{2\le k\\ k\text{ even}}} \left(\hat\phi^{k+1} - \hat\phi^{k-1}\right)\\ &= -\hat\phi^{2-1}\\ &= -\hat\phi^1\\ &= \phi^{-1}.\end{aligned}$$

so that

$$\phi^{-1} - 1 < \sum_{1\le j\le r} \hat\phi^{k_j} \le \phi^{-1}.$$

That is,

$$\begin{aligned}&\phi^{-1} - 1 < \sum_{1\le j\le r} \hat\phi^{k_j} \le \phi^{-1}\\ \iff\quad &\phi^{-1}n + \phi^{-1} - 1 < \phi^{-1}n + \sum_{1\le j\le r} \hat\phi^{k_j} \le \phi^{-1}n + \phi^{-1}\\ \iff\quad &\phi^{-1}n + \phi^{-1} - 1 < \sum_{1\le j\le r} F_{k_j-1} \le \phi^{-1}n + \phi^{-1}\\ \iff\quad &\sum_{1\le j\le r} F_{k_j+1} = \lfloor \phi^{-1}n + \phi^{-1} \rfloor\end{aligned}$$

as we needed to show. □

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[*CMath*, §6.6]

**42.** [*M26*] (D. A. Klarner.) Show that if $m$ and $n$ are nonnegative integers, there is a unique sequence of indices $k_1 \gg k_2 \gg \cdots \gg k_r$ such that

$$m = F_{k_1} + F_{k_2} + \cdots + F_{k_r}, \qquad n = F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_r+1}.$$

(See exercise 34. The $k$'s may be negative, and $r$ may be zero.)

We may prove the existence of such a sequence.

**Proposition.** *There exists a unique sequence of indices $k_j$, where $k_j > k_{j+1} + 1$ for $1 \le j < r$, $r \ge 0$, such that $m = \sum_{1 \le j \le r} F_{k_j}$ and $n = \sum_{1 \le j \le r} F_{k_j+1}$ if $m, n \ge 0$.*

*Proof.* Let $m$ and $n$ be nonnegative integers. We must show that there exists a unique sequence of indices $k_j$, where $k_j > k_{j+1} + 1$ for $1 \le j < r$, $r \ge 0$, such that

$$m = \sum_{1 \le j \le r} F_{k_j}, \qquad n = \sum_{1 \le j \le r} F_{k_j+1}.$$

If such a sequence exists, we must have for all integers $N$,

$$\begin{aligned} mF_{N-1} + nF_N &= \sum_{1 \le j \le r} F_{k_j} F_{N-1} + \sum_{1 \le j \le r} F_{k_j+1} F_N \\ &= \sum_{1 \le j \le r} (F_{k_j} F_{N-1} + F_{k_j+1} F_N) \\ &= \sum_{1 \le j \le r} F_{k_j+N} && \text{by Eq. (6).} \end{aligned}$$

In the trivial case that $r = 0$, the representation is unique: in particular, the empty one. Otherwise, in the case that $r > 0$, let $N = -k_r + 2$ and $k'_j = k_j + N$ for $1 \le j \le r$, so that

$$\begin{aligned} k'_j &= k_j + N \\ &= k_j - k_r + 2, \end{aligned}$$

and since $k_j \ge k_r$, so that

$$k'_j > 1.$$

Then, by exercise 34, the representation

$$\sum_{1 \le j \le r} F_{k'_j}$$

must be unique. Now let $N$ be large enough so that

$$\left| m\hat{\phi}^{N-1} + n\hat{\phi}^N \right| < \phi^{-2}.$$

Since $\phi^2 = \phi + 1$,

$$\begin{aligned} \phi^2 \ge \phi + 1 \quad &\iff \quad \phi \ge \phi^{-1} + 1 \\ &\iff \quad \phi - 1 \ge \phi^{-1} \\ &\iff \quad 1 - \phi \le -\phi^{-1} \\ &\iff \quad \phi^{-1} - 1 \le -\phi^{-2}; \end{aligned}$$

and since $\phi > 1$,

$$\begin{aligned} \phi \ge 1 \quad &\iff \quad \phi \le \phi^2 \\ &\iff \quad \phi^{-2} \le \phi^{-1}; \end{aligned}$$

we have that

$$\begin{aligned}
&\left|m\hat{\phi}^{N-1}+n\hat{\phi}^{N}\right|<\phi^{-2}\\
&\iff\quad -\phi^{-2}<m\hat{\phi}^{N-1}+n\hat{\phi}^{N}<\phi^{-2}\\
&\iff\quad \phi^{-1}-1<m\hat{\phi}^{N-1}+n\hat{\phi}^{N}<\phi^{-2}\\
&\implies\quad \phi^{-1}-1<m\hat{\phi}^{N-1}+n\hat{\phi}^{N}\le\phi^{-1}\\
&\implies\quad \phi\left(mF_{N-1}+nF_N\right)+\phi^{-1}-1\\
&\qquad\quad <\phi\left(mF_{N-1}+nF_N\right)+\left(m\hat{\phi}^{N-1}+n\hat{\phi}^{N}\right)\\
&\qquad\quad \le\phi\left(mF_{N-1}+nF_N\right)+\phi^{-1}\\
&\implies\quad \phi\left(mF_{N-1}+nF_N\right)+\left(m\hat{\phi}^{N-1}+n\hat{\phi}^{N}\right)=\left\lfloor\phi\left(mF_{N-1}+nF_N\right)+\phi^{-1}\right\rfloor.
\end{aligned}$$

Then

$$\begin{aligned}
mF_N+nF_{N+1}&=m\left(\phi F_{N-1}+\hat{\phi}^{N-1}\right)+n\left(\phi F_N+\hat{\phi}^{N}\right) &&\text{by exercise 21}\\
&=m\phi F_{N-1}+m\hat{\phi}^{N-1}+n\phi F_N+n\hat{\phi}^{N}\\
&=\phi\left(mF_{N-1}+nF_N\right)+\left(m\hat{\phi}^{N-1}+n\hat{\phi}^{N}\right)\\
&=\left\lfloor\phi\left(mF_{N-1}+nF_N\right)+\phi^{-1}\right\rfloor\\
&=\sum_{1\le j\le r}F_{k_j+N+1}. &&\text{by exercise 41}
\end{aligned}$$

Finally, setting $N=-1$ yields

$$\begin{aligned}
mF_{-1}+nF_{-1+1}&=m+nF_0\\
&=m+0\\
&=m\\
&=\sum_{1\le j\le r}F_{k_j-1+1}\\
&=\sum_{1\le j\le r}F_{k_j},
\end{aligned}$$

and setting $N=0$ yields

$$\begin{aligned}
mF_0+nF_{0+1}&=0+nF_1\\
&=n\\
&=\sum_{1\le j\le r}F_{k_j+0+1}\\
&=\sum_{1\le j\le r}F_{k_j+1},
\end{aligned}$$

concluding our proof that there exists a unique sequence of indices $k_j$, where $k_j > k_{j+1}+1$ for $1\le j<r$, $r\ge 0$, such that

$$m=\sum_{1\le j\le r}F_{k_j},\qquad n=\sum_{1\le j\le r}F_{k_j+1},$$

as we needed to show. □

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[D. A. Klarner, *Fibonacci Quarterly* **6** (1968), 235–244]