Exercises from Section 1.2.9

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1. [*M12*] What is the generating function for the sequence $2, 5, 13, 35, \ldots = \langle 2^n + 3^n \rangle$?

Let $G(z) = \sum_{n\geq 0} (2^n + 3^n)\, z^n$ be the generating function for the sequence $\langle 2^n + 3^n \rangle$. Then

$$\begin{aligned} G(z) &= \sum_{n\geq 0} (2^n + 3^n)\, z^n \\ &= \sum_{n\geq 0} 2^n z^n + \sum_{n\geq 0} 3^n z^n && \text{by Eq. (2)} \\ &= \sum_{n\geq 0} (2z)^n + \sum_{n\geq 0} (3z)^n \\ &= \frac{1}{1-2z} + \frac{1}{1-3z}. && \text{by Eq. (5)} \end{aligned}$$

► **2.** [*M13*] Prove Eq. (11).

Proposition. $\left(\sum_{n\geq 0} \frac{a_n}{n!} z^n\right)\left(\sum_{n\geq 0} \frac{b_n}{n!} z^n\right) = \sum_{n\geq 0} \frac{\sum_{0\leq k\leq n} \binom{n}{k} a_k b_{n-k}}{n!} z^n.$

Proof. Let n be an arbitrary nonnegative integer, and $\langle a_n/n! \rangle$, $\langle b_n/n! \rangle$ two arbitrary sequences. We must show that

$$\left(\sum_{n\geq 0} \frac{a_n}{n!} z^n\right)\left(\sum_{n\geq 0} \frac{b_n}{n!} z^n\right) = \sum_{n\geq 0} \frac{\sum_{0\leq k\leq n} \binom{n}{k} a_k b_{n-k}}{n!} z^n.$$

But

$$\begin{aligned} \left(\sum_{n\geq 0} \frac{a_n}{n!} z^n\right)\left(\sum_{n\geq 0} \frac{b_n}{n!} z^n\right) &= \sum_{n\geq 0} \sum_{0\leq k\leq n} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!} z^n && \text{by Eq. (6)} \\ &= \sum_{n\geq 0} \sum_{0\leq k\leq n} \frac{1}{n!} \frac{n!}{k!(n-k)!} a_k b_{n-k} z^n \\ &= \sum_{n\geq 0} \sum_{0\leq k\leq n} \frac{1}{n!} \binom{n}{k} a_k b_{n-k} z^n \\ &= \sum_{n\geq 0} \frac{\sum_{0\leq k\leq n} \binom{n}{k} a_k b_{n-k}}{n!} z^n \end{aligned}$$

as we needed to show. □

3. [*HM21*] Differentiate the generating function (18) for $\langle H_n \rangle$, and compare this with the generating function for $\langle \sum_{k=0}^{n} H_k \rangle$. What relation can you deduce?

If we differentiate the generating function for the harmonic numbers $\langle H_n \rangle$, Eq. (18),

$$G(z) = \sum_{n \geq 0} H_n z^n = \frac{1}{1-z} \ln \frac{1}{1-z},$$

we find that

$$\begin{aligned}
\frac{d}{dz} G(z) &= \frac{d}{dz} \left(\frac{1}{1-z} \ln \frac{1}{1-z} \right) \\
&= \frac{d}{dz} \left(\frac{1}{1-z} \right) \ln \frac{1}{1-z} + \frac{1}{1-z} \frac{d}{dz} \left(\ln \frac{1}{1-z} \right) \\
&= \frac{1}{(1-z)^2} \ln \frac{1}{1-z} + \frac{1}{1-z} \frac{d}{dz} \left(\ln \frac{1}{1-z} \right) && \text{by Eq. (16)} \\
&= \frac{1}{(1-z)^2} \ln \frac{1}{1-z} + \frac{1}{1-z} \frac{\frac{d}{dz}\left(1/(1-z)\right)}{1/(1-z)} \\
&= \frac{1}{(1-z)^2} \ln \frac{1}{1-z} + \frac{1-z}{1-z} \frac{1}{(1-z)^2} && \text{by Eq. (16)} \\
&= \frac{1}{(1-z)^2} \ln \frac{1}{1-z} + \frac{1}{(1-z)^2}.
\end{aligned}$$

The generating function for $\langle\sum_{0\le k\le n} H_k\rangle$ is

$$
\begin{aligned}
H(z) &= \sum_{n\ge 0}\sum_{0\le k\le n} H_k z^n \\
&= \sum_{n\ge 0}\left(\sum_{0\le k\le n} H_k\cdot 1\right) z^n \\
&= \left(\sum_{n\ge 0} H_n z^n\right)\left(\sum_{n\ge 0} 1\cdot z^n\right) && \text{by Eq. (6)} \\
&= \left(\sum_{n\ge 0} H_n z^n\right)\left(\sum_{n\ge 0} z^n\right) \\
&= G(z)\frac{1}{1-z} && \text{by Eq. (5)} \\
&= \frac{1}{1-z}\ln\left(\frac{1}{1-z}\right)\frac{1}{1-z} \\
&= \frac{1}{(1-z)^2}\ln\frac{1}{1-z} \\
&= \frac{d}{dz}G(z) - \frac{1}{(1-z)^2} \\
&= \frac{d}{dz}G(z) - \frac{d}{dz}\frac{1}{1-z} && \text{by Eq. (16)} \\
&= \frac{d}{dz}G(z) - \sum_{n\ge 0}(n+1)z^n && \text{by Eq. (16)} \\
&= \sum_{n\ge 0}(n+1)H_{n+1}z^n - \sum_{n\ge 0}(n+1)z^n && \text{by Eq. (14)} \\
&= \sum_{n\ge 0}\left((n+1)H_{n+1} - (n+1)\right) z^n \\
&= \sum_{n\ge 0}\left((n+1)H_{n+1} - 1 - n\right) z^n \\
&= \sum_{n\ge 0}\left((n+1)H_{n+1} - \frac{n+1}{n+1} - n\right) z^n \\
&= \sum_{n\ge 0}\left((n+1)\left(H_{n+1} - \frac{1}{n+1}\right) - n\right) z^n \\
&= \sum_{n\ge 0}\left((n+1)H_n - n\right) z^n \\
&= \sum_{n\ge 0}\left(nH_n + H_n - n\right) z^n.
\end{aligned}
$$

Then,

$$\begin{aligned}
&\sum_{0\le k\le n} H_k = nH_n + H_n - n \\
&\iff \quad \sum_{0\le k\le n} H_k - H_n = nH_n - n \\
&\iff \quad \sum_{0\le k\le n-1} H_k = nH_n - n \\
&\iff \quad \sum_{0\le k\le n-1} H_k - 0 = nH_n - n \\
&\iff \quad \sum_{0\le k\le n-1} H_k - H_0 = nH_n - n \\
&\iff \quad \sum_{1\le k\le n-1} H_k = nH_n - n,
\end{aligned}$$

in agreement with Eq. 1.2.7-(8).

4. [*M01*] Explain why Eq. (19) is a special case of Eq. (21).

Eq. (19),

$$(1+z)^r = \sum_{k\ge 0} \binom{r}{k} z^k,$$

is a special case of Eq. (21),

$$x^r = \sum_{k\ge 0} \binom{r-kt}{k} \frac{r}{r-kt} z^k,$$

for $t = 0$. Then, since $x^{t+1} = x^t + z = x = 1 + z$,

$$\begin{aligned}
x^r &= (1+z)^r \\
&= \sum_{k\ge 0} \binom{r-kt}{k} \frac{r}{r-kt} z^k \\
&= \sum_{k\ge 0} \binom{r}{k} \frac{r}{r} z^k \\
&= \sum_{k\ge 0} \binom{r}{k} z^k.
\end{aligned}$$

5. [*M20*] Prove Eq. (23) by induction on n.

Proposition. $(e^z - 1)^n = n! \sum_k \left\{ {k \atop n} \right\} \frac{z^k}{k!}$.

Proof. Let n be an arbitrary nonnegative integer. We must show that

$$(e^z - 1)^n = n! \sum_k \left\{ {k \atop n} \right\} \frac{z^k}{k!}.$$

If $n = 0$,

$$\begin{aligned}(e^z - 1)^0 &= 1 \\ &= z^0 \\ &= \left\{ {0 \atop 0} \right\} \frac{z^0}{0!} \\ &= \sum_k \left\{ {k \atop 0} \right\} \frac{z^k}{k!} \\ &= 0! \sum_k \left\{ {k \atop 0} \right\} \frac{z^k}{k!}.\end{aligned}$$

Then, assuming

$$(e^z - 1)^n = n! \sum_k \left\{ {k \atop n} \right\} \frac{z^k}{k!},$$

we must show

$$(e^z - 1)^{n+1} = (n+1)! \sum_k \left\{ {k \atop n+1} \right\} \frac{z^k}{k!}.$$

But

$$\begin{aligned}(e^z - 1)^{n+1} &= (e^z - 1)(e^z - 1)^n \\ &= (e^z - 1) n! \sum_k \left\{ {k \atop n} \right\} \frac{z^k}{k!} \\ &= \left(\left(\sum_{k \geq 0} \frac{1}{k!} z^k \right) - 1 \right) n! \sum_k \left\{ {k \atop n} \right\} \frac{z^k}{k!} && \text{by Eq. (22)} \\ &= \left(\sum_{k \geq 0} \frac{1}{k!} z^k \right) \left(n! \sum_k \left\{ {k \atop n} \right\} \frac{z^k}{k!} \right) - n! \sum_k \left\{ {k \atop n} \right\} \frac{z^k}{k!} \\ &= n! \left(\sum_{k \geq 0} \frac{1}{k!} z^k \right) \left(\sum_{k \geq 0} \left\{ {k \atop n} \right\} \frac{1}{k!} z^k \right) - n! \sum_k \left\{ {k \atop n} \right\} \frac{z^k}{k!} \\ &= n! \sum_k \frac{1}{k!} \sum_j \binom{k}{j} \left\{ {j \atop n} \right\} z^k - n! \sum_k \left\{ {k \atop n} \right\} \frac{z^k}{k!} && \text{by Eq. (11)} \\ &= n! \sum_k \frac{1}{k!} \left\{ {k+1 \atop n+1} \right\} z^k - n! \sum_k \left\{ {k \atop n} \right\} \frac{z^k}{k!} && \text{by Eq. 1.2.6-(52)} \\ &= n! \sum_k \left(\left\{ {k+1 \atop n+1} \right\} - \left\{ {k \atop n} \right\} \right) \frac{z^k}{k!} \\ &= n! \sum_k (n+1) \left\{ {k \atop n+1} \right\} \frac{z^k}{k!} && \text{by Eq. 1.2.6-(46)} \\ &= (n+1)! \sum_k \left\{ {k \atop n+1} \right\} \frac{z^k}{k!}\end{aligned}$$

as we needed to show. □

▶ **6.** [*HM15*] Find the generating function for

$$\langle \sum_{0<k<n} \frac{1}{k(n-k)} \rangle;$$

differentiate it and express the coefficients in terms of harmonic numbers.

The generating function for

$$\langle \sum_{0<k<n} \frac{1}{k(n-k)} \rangle$$

is

$$\begin{aligned}
G(z) &= \sum_{n>0} \sum_{0<k<n} \frac{1}{k(n-k)} z^n \\
&= \sum_{n>0} \sum_{0<k<n} \frac{1}{k} \frac{1}{n-k} z^n \\
&= \left(\sum_{n>0} \frac{1}{n} z^n \right) \left(\sum_{n>0} \frac{1}{n} z^n \right) && \text{by Eq. (6)} \\
&= \left(\sum_{n>0} \frac{1}{n} z^n \right)^2 \\
&= \left(\ln \frac{1}{1-z} \right)^2. && \text{by Eq. (17)}
\end{aligned}$$

Differentiating, we find

$$\begin{aligned}
\frac{d}{dz} G(z) &= \frac{d}{dz} \left(\ln \frac{1}{1-z} \right)^2 \\
&= 2 \ln \frac{1}{1-z} \frac{d}{dz} \ln \frac{1}{1-z} \\
&= 2 \ln \frac{1}{1-z} (1-z) \frac{d}{dz} frac11-z \\
&= 2 \ln \frac{1}{1-z} (1-z) \frac{1}{(1-z)^2} && \text{by Eq. (16)} \\
&= 2 \frac{1}{1-z} \ln \frac{1}{1-z} \\
&= 2 \sum_{n \geq 0} H_n z^n. && \text{by Eq. (18)}
\end{aligned}$$

Then,

$$\begin{aligned}
G(z) &= \int_0^z \frac{d}{dt} G(t) dt \\
&= \int_0^z 2 \sum_{n \geq 0} H_n t^n dt \\
&= 2 \int_0^z \sum_{n \geq 0} H_n t^n dt \\
&= 2 \sum_{n \geq 1} \frac{1}{n} H_{n-1} z^n && \text{by Eq. (15)} \\
&= \sum_{n>0} \frac{2}{n} H_{n-1} z^n.
\end{aligned}$$

And so,

$$\sum_{0<k<n} \frac{1}{k(n-k)} = 2H_{n-1}/n$$

for $n > 0$.

7. [*M15*] Verify all the steps leading to Eq. (38).

Suppose that we have n numbers $x_1, x_2, \cdots, x_n$ and we want the sum

$$h_m = \sum_{1 \le j_1 \le n} x_{j_1} \sum_{j_1 \le j_2 \le n} x_{j_2} \cdots \sum_{j_{m-1} \le j_m \le n} x_{j_m} = \sum_{1 \le j_1 \le \cdots \le j_m \le n} x_{j_1} \cdots x_{j_m},$$

expressed in terms of $S_1, S_2, \cdots, S_m$, where

$$S_j = \sum_{1 \le k \le n} x_k^j,$$

the sum of jth powers. First, we establish the identity

$$\sum_{1 \le j_1 \le \cdots \le j_m \le n} x_{j_1} \cdots x_{j_m} = \sum_{\substack{k_1, k_2, \ldots, k_n \ge 0 \\ k_1 + k_2 + \cdots + k_n = n}} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}. \tag{7.1}$$

If $n = 1$,

$$\begin{aligned} \sum_{1 \le j_1 \le 1} x_{j_1} &= x_1 \\ &= x_1^1 \\ &= \sum_{\substack{k_1 \ge 0 \\ k_1 = 1}} x_1^{k_1}. \end{aligned}$$

Then, assuming

$$\sum_{1 \le j_1 \le \cdots \le j_m \le n} x_{j_1} \cdots x_{j_m} = \sum_{\substack{k_1, k_2, \ldots, k_n \ge 0 \\ k_1 + k_2 + \cdots + k_n = n}} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n},$$

we must show

$$\sum_{1 \le j_1 \le \cdots \le j_{m+1} \le n+1} x_{j_1} \cdots x_{j_{m+1}} = \sum_{\substack{k_1, k_2, \ldots, k_{n+1} \ge 0 \\ k_1 + k_2 + \cdots + k_{n+1} = n+1}} x_1^{k_1} x_2^{k_2} \cdots x_{n+1}^{k_{n+1}}.$$

But

$$\begin{aligned} \sum_{1 \le j_1 \le \cdots \le j_{m+1} \le n+1} x_{j_1} \cdots x_{j_{m+1}} &= \sum_{1 \le j_{m+1} \le n+1} x_{j_{m+1}} \sum_{1 \le j_1 \le \cdots \le j_m \le j_{m+1}} x_{j_1} \cdots x_{j_m} \\ &= \sum_{1 \le j_{m+1} \le n+1} x_{j_{m+1}} \sum_{\substack{k_1, k_2, \ldots, k_n \ge 0 \\ k_1 + k_2 + \cdots + k_n = n}} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \\ &= \sum_{\substack{k_1+1, k_2, \ldots, k_n, 0 \ge 0 \\ k_1 + 1 + k_2 + \cdots + k_n + 0 = n+1}} x_1^{k_1+1} x_2^{k_2} \cdots x_n^{k_n} x_{n+1}^0 \\ &\quad + \sum_{\substack{k_1, k_2+1, \ldots, k_n, 0 \ge 0 \\ k_1 + k_2 + 1 + \cdots + k_n + 0 = n+1}} x_1^{k_1} x_2^{k_2+1} \cdots x_n^{k_n} x_{n+1}^0 \\ &\quad + \cdots + \sum_{\substack{k_1, k_2, \ldots, k_n+1, 0 \ge 0 \\ k_1 + k_2 + \cdots + k_n + 1 = n+1}} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n+1} x_{n+1}^0 \\ &\quad + \sum_{\substack{k_1, k_2, \ldots, k_n, 1 \ge 0 \\ k_1 + k_2 + \cdots + k_n + 1 = n+1}} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} x_{n+1}^1 \\ &= \sum_{\substack{k_1, k_2, \ldots, k_{n+1} \ge 0 \\ k_1 + k_2 + \cdots + k_{n+1} = n+1}} x_1^{k_1} x_2^{k_2} \cdots x_{n+1}^{k_{n+1}}, \end{aligned}$$

as we needed to show. Now let

$$G(z) = \sum_{k \geq 0} h_k z^k.$$

Then

$$\begin{aligned}
G(z) &= \sum_{k \geq 0} h_k z^k \\
&= \sum_{k \geq 0} \sum_{1 \leq j_1 \leq \cdots \leq j_k \leq n} x_{j_1} \cdots x_{j_k} z^k \\
&= \sum_{k \geq 0} z^k \sum_{1 \leq j_1 \leq \cdots \leq j_k \leq n} x_{j_1} \cdots x_{j_k} \\
&= \sum_{k \geq 0} z^k \sum_{\substack{k_1, k_2, \ldots, k_n \geq 0 \\ k_1 + k_2 + \cdots + k_n = n}} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} && \text{by (7.1)} \\
&= \prod_{1 \leq j \leq n} \sum_{k \geq 0} x_j^k z^k && \text{by Eq. (9)} \\
&= \prod_{1 \leq j \leq n} \sum_{k \geq 0} (x_j z)^k \\
&= \prod_{1 \leq j \leq n} \frac{1}{1 - x_j z} && \text{by Eq. (5).}
\end{aligned}$$

Taking the logarithm yields

$$\begin{aligned}
\ln G(z) &= \ln \prod_{1 \leq j \leq n} \frac{1}{1 - x_j z} \\
&= \sum_{1 \leq j \leq n} \ln \frac{1}{1 - x_j z} \\
&= \sum_{1 \leq j \leq n} \sum_{k \geq 1} \frac{1}{k} (x_j z)^k && \text{by Eq. (17)} \\
&= \sum_{k \geq 1} \frac{\sum_{1 \leq j \leq n} x_j^k z^k}{k} \\
&= \sum_{k \geq 1} \frac{S_k z^k}{k}.
\end{aligned}$$

Finally,

$$
\begin{aligned}
G(z) &= e^{\ln G(z)} \\
&= \exp\left(\sum_{k\geq 1} \frac{S_k z^k}{k}\right) \\
&= \prod_{k\geq 1} e^{S_k z^k/k} \\
&= \prod_{k\geq 1} \sum_{j\geq 0} \frac{1}{j!}\left(\frac{S_k z^k}{k}\right)^j && \text{by Eq. (22)} \\
&= \prod_{k\geq 1} \sum_{j\geq 0} \frac{S_k^j \left(z^k\right)^j}{k^j j!} \\
&= \prod_{j\geq 1} \sum_{k\geq 0} \frac{S_j^k}{j^k k!}\left(z^j\right)^k \\
&= \prod_{j\geq 1} \sum_{k\geq 0} \frac{S_j^k}{j^k k!} z^{jk} \\
&= \prod_{j\geq 1} \sum_{k/j\geq 0} \frac{S_j^{k/j}}{j^{k/j}(k/j)!} z^k \\
&= \sum_{n\geq 0} z^n \sum_{\substack{k_1,k_2,\ldots,k_n\geq 0 \\ k_1+k_2+\cdots+k_n=n}} \frac{S_1^{k_1/1}}{1^{k_1/1}(k_1/1)!}\frac{S_2^{k_2/2}}{2^{k_2/2}(k_2/2)!}\cdots\frac{S_n^{k_n/n}}{n^{k_n/n}(k_n/n)!} && \text{by Eq. (9)} \\
&= \sum_{n\geq 0} z^n \sum_{\substack{k_1,k_2,\ldots,k_n\geq 0 \\ k_1+2k_2+\cdots+nk_n=n}} \frac{S_1^{k_1}}{1^{k_1}k_1!}\frac{S_2^{k_2}}{2^{k_2}k_2!}\cdots\frac{S_n^{k_n}}{n^{k_n}k_n!},
\end{aligned}
$$

and hence Eq. (38).

8. [*M23*] Find the generating function for $p(n)$, the number of partitions of n.

The number of partitions $p(n)$ corresponds to the number of solutions of the equation $k_1 + 2k_2 +$

$\cdots + nk_n = n$, where $k_j \geq 0$ is the number of js in the partition. For example, for $n = 5$,

$$\begin{aligned}
p(5) &= \sum_{\substack{k_1, k_2, \cdots, k_5 \geq 0 \\ k_1 + 2k_2 + \cdots + 5k_5 = 5}} 1 \\
&= (1 \cdot 5 + 2 \cdot 0 + 3 \cdot 0 + 4 \cdot 0 + 5 \cdot 0)/5 \\
&\quad + (1 \cdot 3 + 2 \cdot 1 + 3 \cdot 0 + 4 \cdot 0 + 5 \cdot 0)/5 \\
&\quad + (1 \cdot 1 + 2 \cdot 2 + 3 \cdot 0 + 4 \cdot 0 + 5 \cdot 0)/5 \\
&\quad + (1 \cdot 2 + 2 \cdot 0 + 3 \cdot 1 + 4 \cdot 0 + 5 \cdot 0)/5 \\
&\quad + (1 \cdot 0 + 2 \cdot 1 + 3 \cdot 1 + 4 \cdot 0 + 5 \cdot 0)/5 \\
&\quad + (1 \cdot 1 + 2 \cdot 0 + 3 \cdot 0 + 4 \cdot 1 + 5 \cdot 0)/5 \\
&\quad + (1 \cdot 0 + 2 \cdot 0 + 3 \cdot 0 + 4 \cdot 1 + 5 \cdot 1)/5 \\
&= (1 + 1 + 1 + 1 + 1)/5 \\
&\quad + (1 + 1 + 1 + 2)/5 \\
&\quad + (1 + 2 + 2)/5 \\
&\quad + (1 + 1 + 3)/5 \\
&\quad + (2 + 3)/5 \\
&\quad + (1 + 4)/5 \\
&\quad + (5)/5 \\
&= 7.
\end{aligned}$$

The generating function for $p(n)$ is therefore obtained by making all possible combinations from 1 to n available through multiplication, so that the coefficient is the sum count of these combinations, as

$$G(z) = \sum_{n \geq 0} p(n) z^n = \prod_{j \geq 1} \sum_{k \geq 0} z^{jk}.$$

For example, for $n = 5$,

$$\begin{aligned}
[z^5]G(z) &= [z^5]\sum_{n\ge 0} p(n)z^n \\
&= [z^5]\prod_{j\ge 1}\sum_{k\ge 0} z^{jk} \\
&= [z^5]\cdots + z^{1\cdot 5+2\cdot 0+3\cdot 0+4\cdot 0+5\cdot 0} \\
&\quad + z^{1\cdot 3+2\cdot 1+3\cdot 0+4\cdot 0+5\cdot 0} \\
&\quad + z^{1\cdot 1+2\cdot 2+3\cdot 0+4\cdot 0+5\cdot 0} \\
&\quad + z^{1\cdot 2+2\cdot 0+3\cdot 1+4\cdot 0+5\cdot 0} \\
&\quad + z^{1\cdot 0+2\cdot 1+3\cdot 1+4\cdot 0+5\cdot 0} \\
&\quad + z^{1\cdot 1+2\cdot 0+3\cdot 0+4\cdot 1+5\cdot 0} \\
&\quad + z^{1\cdot 0+2\cdot 0+3\cdot 0+4\cdot 1+5\cdot 1} + \cdots \\
&= [z^5]\cdots + z^{1+1+1+1+1} \\
&\quad + z^{1+1+1+2} \\
&\quad + z^{1+2+2} \\
&\quad + z^{1+1+3} \\
&\quad + z^{2+3} \\
&\quad + z^{1+4} \\
&\quad + z^{5} + \cdots \\
&= [z^5]7z^5 \\
&= 7.
\end{aligned}$$

So,

$$\begin{aligned}
G(z) &= \sum_{n\ge 0}\sum_{\substack{k_1,k_2,\cdots,k_n\ge 0\\ k_1+2k_2+\cdots+nk_n=n}} z^n \\
&= \sum_{n\ge 0} z^n \sum_{\substack{k_1,k_2,\cdots,k_n\ge 0\\ k_1+2k_2+\cdots+nk_n=n}} 1^{k_1}1^{k_2}\cdots 1^{k_n} \\
&= \sum_{n\ge 0} z^n \sum_{\substack{k_1,k_2,\cdots,k_n\ge 0\\ k_1+k_2+\cdots+k_n=n}} 1^{k_1/1}1^{k_2/2}\cdots 1^{k_n/n} \\
&= \prod_{j\ge 1}\sum_{k/j\ge 0} 1^{k/j} z^k && \text{by Eq. (9)} \\
&= \prod_{j\ge 1}\sum_{k\ge 0} z^{jk} \\
&= \prod_{j\ge 1}\sum_{k\ge 0} \left(z^j\right)^k \\
&= \prod_{j\ge 1}\frac{1}{1-z^j} && \text{by Eq. (5)} \\
&= \frac{1}{\prod_{n\ge 1}(1-z^n)}.
\end{aligned}$$

That is,

$$\begin{aligned} G(z) &= \sum_{n\ge 0} p(n) z^n \\ &= \sum_{n\ge 0} \sum_{\substack{k_1,k_2,\cdots,k_n\ge 0\\ k_1+2k_2+\cdots+nk_n=n}} z^n \\ &= \prod_{j\ge 1}\sum_{k\ge 0} z^{jk} \\ &= \frac{1}{\prod_{n\ge 1}(1-z^n)}. \end{aligned}$$

[G. Pólya, *Induction and Analogy in Mathematics* (Princeton: Princeton University Press, 1954), Chapter 6]

9. [*M11*] In the notation of Eqs. (34) and (35), what is h_4 in terms of S_1, S_2, S_3, and S_4?

Suppose that we have n numbers $x_1, x_2, \cdots, x_n$ and we want the sum

$$h_m = \sum_{1\le j_1\le n} x_{j_1} \sum_{j_1\le j_2\le n} x_{j_2} \cdots \sum_{j_{m-1}\le j_m\le n} x_{j_m} = \sum_{1\le j_1\le\cdots\le j_m\le n} x_{j_1}\cdots x_{j_m},$$

expressed in terms of $S_1, S_2, \cdots, S_m$, where

$$S_j = \sum_{1\le k\le n} x_k^j,$$

the sum of jth powers. In particular, h_4. But

$$\begin{aligned} [z^4]G(z) &= [z^4]\sum_{k\ge 0} h_k z^k && \text{by Eq. (35)} \\ &= [z^4]\sum_{n\ge 0} z^n \sum_{\substack{k_1,k_2,\ldots,k_n\ge 0\\ k_1+2k_2+\cdots+nk_n=n}} \frac{S_1^{k_1}}{1^{k_1}k_1!}\frac{S_2^{k_2}}{2^{k_2}k_2!}\cdots\frac{S_n^{k_n}}{n^{k_n}k_n!} && \text{by Eq. (38)} \\ &= \sum_{\substack{k_1,k_2,k_3,k_4\ge 0\\ k_1+2k_2+3k_3+4k_4=4}} \frac{S_1^{k_1}}{1^{k_1}k_1!}\frac{S_2^{k_2}}{2^{k_2}k_2!}\frac{S_3^{k_3}}{3^{k_3}k_3!}\frac{S_4^{k_4}}{4^{k_n}k_4!} \\ &= \frac{S_1^4}{1^4 4!}\frac{S_2^0}{2^0 0!}\frac{S_3^0}{3^0 0!}\frac{S_4^0}{4^0 0!} \\ &\quad + \frac{S_1^2}{1^2 2!}\frac{S_2^1}{2^1 1!}\frac{S_3^0}{3^0 0!}\frac{S_4^0}{4^0 0!} \\ &\quad + \frac{S_1^1}{1^1 1!}\frac{S_2^0}{2^0 0!}\frac{S_3^1}{3^1 1!}\frac{S_4^0}{4^0 0!} \\ &\quad + \frac{S_1^0}{1^0 0!}\frac{S_2^2}{2^2 2!}\frac{S_3^0}{3^0 0!}\frac{S_4^0}{4^0 0!} \\ &\quad + \frac{S_1^0}{1^0 0!}\frac{S_2^0}{2^0 0!}\frac{S_3^0}{3^0 0!}\frac{S_4^1}{4^1 1!} \\ &= \frac{S_1^4}{1^4 4!} + \frac{S_1^2}{1^2 2!}\frac{S_2^1}{2^1 1!} + \frac{S_1^1}{1^1 1!}\frac{S_3^1}{3^1 1!} + \frac{S_2^2}{2^2 2!} + \frac{S_4^1}{4^1 1!} \\ &= \frac{S_1^4}{24} + \frac{S_1^2}{2}\frac{S_2}{2} + S_1\frac{S_3}{3} + \frac{S_2^2}{8} + \frac{S_4}{4} \\ &= \frac{1}{24}S_1^4 + \frac{1}{4}S_1^2 S_2 + \frac{1}{8}S_2^2 + \frac{1}{3}S_1 S_3 + \frac{1}{4}S_4. \end{aligned}$$

▶ **10.** [*M25*] An *elementary symmetric function* is defined by the formula

$$e_m = \sum_{1 \le j_1 < \cdots < j_m \le n} x_{j_1} \ldots x_{j_m}.$$

(This is the same as h_m of Eq. (33), except that equal subscripts are not allowed.) Find the generating function for e_m, and express e_m in terms of the S_j in Eq. (34). Write out the formulas for e_1, e_2, e_3, and e_4.

Suppose that we have n numbers $x_1, x_2, \cdots, x_n$ and we want the elementary symmetric function

$$e_m = \sum_{1 \le j_1 \le n} x_{j_1} \sum_{j_1 < j_2 \le n} x_{j_2} \cdots \sum_{j_{m-1} < j_m \le n} x_{j_m} = \sum_{1 \le j_1 < \cdots < j_m \le n} x_{j_1} \cdots x_{j_m},$$

expressed in terms of $S_1, S_2, \cdots, S_m$, where

$$S_j = \sum_{1 \le k \le n} x_k^j,$$

the sum of jth powers. First, we establish the identity

$$\sum_{k \ge 0} \sum_{1 \le j_1 < \cdots < j_k \le n} x_{j_1} \cdots x_{j_k} z^k = \prod_{1 \le j \le n} (1 + x_j z)\,. \tag{10.1}$$

If $n = 1$,

$$\begin{aligned} &\sum_{k \ge 0} \sum_{1 \le j_1 \le 1} x_{j_1} \cdots x_{j_k} z^k \\ &\qquad = \sum_{k \ge 0} x_1 \cdots x_{j_k} z^k \\ &\qquad = z^0 + x_1 z^1 \\ &\qquad = 1 + x_1 z \\ &\qquad = \prod_{1 \le j \le 1} (1 + x_j z)\,. \end{aligned}$$

Then, assuming

$$\sum_{k \ge 0} \sum_{1 \le j_1 < \cdots < j_k \le n} x_{j_1} \cdots x_{j_k} z^k = \prod_{1 \le j \le n} (1 + x_j z)\,,$$

we must show

$$\sum_{k \ge 0} \sum_{1 \le j_1 < \cdots < j_{k+1} \le n+1} x_{j_1} \cdots x_{j_{k+1}} z^{k+1} = \prod_{1 \le j \le n+1} (1 + x_j z)\,.$$

But

$$\begin{aligned}
&\sum_{k\geq 0}\sum_{1\leq j_1<\cdots<j_{k+1}\leq n+1} x_{j_1}\cdots x_{j_{k+1}} z^{k+1}\\
&\quad=\sum_{k\geq 0}\sum_{1\leq j_{k+1}\leq n+1} x_{j_{k+1}} z \sum_{1\leq j_1<\cdots<j_k<j_{k+1}} x_{j_1}\cdots x_{j_k} z^k\\
&\quad=\sum_{k\geq 0}\sum_{1\leq j_{k+1}<n+1} x_{j_{k+1}} z \sum_{1\leq j_1<\cdots<j_k<j_{k+1}} x_{j_1}\cdots x_{j_k} z^k\\
&\qquad+\sum_{k\geq 0}\sum_{j_{k+1}=n+1} x_{j_{k+1}} z \sum_{1\leq j_1<\cdots<j_k<j_{k+1}} x_{j_1}\cdots x_{j_k} z^k\\
&\quad=\sum_{k\geq 0}\sum_{1\leq j_1<\cdots<j_k\leq n} x_{j_1}\cdots x_{j_k} z^k+\sum_{k\geq 0} x_{n+1} z \sum_{1\leq j_1<\cdots<j_k\leq n} x_{j_1}\cdots x_{j_k} z^k\\
&\quad=(1+x_{n+1}z)\sum_{k\geq 0}\sum_{1\leq j_1<\cdots<j_k\leq n} x_{j_1}\cdots x_{j_k} z^k\\
&\quad=(1+x_{n+1}z)\prod_{1\leq j\leq n}(1+x_j z)\\
&\quad=\prod_{1\leq j\leq n+1}(1+x_j z)\,,
\end{aligned}$$

as we needed to show. Now let

$$G(z)=\sum_{k\geq 0} e_k z^k.$$

Then

$$\begin{aligned}
G(z)&=\sum_{k\geq 0} e_k z^k\\
&=\sum_{k\geq 0}\sum_{1\leq j_1<\cdots<j_k\leq n} x_{j_1}\cdots x_{j_k} z^k\\
&=\prod_{1\leq j\leq n}(1+x_j z)\,. && \text{by (10.1)}
\end{aligned}$$

Taking the logarithm yields

$$\begin{aligned}
\ln G(z)&=\ln\prod_{1\leq j\leq n}(1+x_j z)\\
&=\sum_{1\leq j\leq n}\ln(1+x_j z)\\
&=\sum_{1\leq j\leq n}\sum_{k\geq 1}\frac{(-1)^{k+1}}{k}(x_j z)^k && \text{by Eq. (24)}\\
&=\sum_{k\geq 1}\frac{(-1)^{k+1}\sum_{1\leq j\leq n} x_j^k z^k}{k}\\
&=\sum_{k\geq 1}\frac{(-1)^{k+1} S_k z^k}{k}.
\end{aligned}$$

Finally,

$$
\begin{aligned}
G(z) &= e^{\ln G(z)} \\
&= \exp\left(\sum_{k\ge1}\frac{(-1)^{k+1}S_k z^k}{k}\right) \\
&= \prod_{k\ge1} e^{(-1)^{k+1}S_k z^k/k} \\
&= \prod_{k\ge1}\sum_{j\ge0}\frac{1}{j!}\left(\frac{(-1)^{k+1}S_k z^k}{k}\right)^j && \text{by Eq. (22)} \\
&= \prod_{k\ge1}\sum_{j\ge0}\frac{\left((-1)^{k+1}S_k\right)^j\left(z^k\right)^j}{k^j j!} \\
&= \prod_{j\ge1}\sum_{k\ge0}\frac{\left((-1)^{j+1}S_j\right)^k}{j^k k!}\left(z^j\right)^k \\
&= \prod_{j\ge1}\sum_{k\ge0}\frac{\left((-1)^{j+1}S_j\right)^k}{j^k k!} z^{jk} \\
&= \prod_{j\ge1}\sum_{k/j\ge0}\frac{\left((-1)^{j+1}S_j\right)^{k/j}}{j^{k/j}(k/j)!} z^{k} \\
&= \sum_{n\ge0} z^n \sum_{\substack{k_1,k_2,\ldots,k_n\ge0\\ k_1+k_2+\cdots+k_n=n}} \frac{\left((-1)^{1+1}S_1\right)^{k_1/1}}{1^{k_1/1}(k_1/1)!}\frac{\left((-1)^{2+1}S_2\right)^{k_2/2}}{2^{k_2/2}(k_2/2)!} && \text{by Eq. (9)} \\
&\qquad \cdots\frac{\left((-1)^{n+1}S_n\right)^{k_n/n}}{n^{k_n/n}(k_n/n)!} \\
&= \sum_{n\ge0} z^n \sum_{\substack{k_1,k_2,\ldots,k_n\ge0\\ k_1+2k_2+\cdots+nk_n=n}} \frac{S_1^{k_1}}{1^{k_1}k_1!}\frac{(-S_2)^{k_2}}{2^{k_2}k_2!}\cdots\frac{\left((-1)^{n+1}S_n\right)^{k_n}}{n^{k_n}k_n!},
\end{aligned}
$$

and hence

$$
\begin{aligned}
G(z) &= \sum_{k\ge0} e_k z^k \\
&= \sum_{k\ge0}\sum_{1\le j_1<\cdots<j_m\le n} x_{j_1}\cdots x_{j_m} z^k \\
&= \sum_{n\ge0} z^n \sum_{\substack{k_1,k_2,\ldots,k_n\ge0\\ k_1+2k_2+\cdots+nk_n=n}} \frac{S_1^{k_1}}{1^{k_1}k_1!}\frac{(-S_2)^{k_2}}{2^{k_2}k_2!}\cdots\frac{\left((-1)^{n+1}S_n\right)^{k_n}}{n^{k_n}k_n!}.
\end{aligned}
$$

We then have

$$
\begin{aligned}
e_1 &= \sum_{\substack{k_1\ge0\\ k_1=1}} \frac{S_1^{k_1}}{1^{k_1}k_1!} \\
&= \frac{S_1^1}{1^1 1!} \\
&= S_1,
\end{aligned}
$$

$$\begin{aligned} e_2 &= \sum_{\substack{k_1,k_2\ge 0\\ k_1+2k_2=2}} \frac{S_1^{k_1}}{1^{k_1}k_1!}\frac{(-S_2)^{k_2}}{2^{k_2}k_2!} \\ &= \frac{S_1^2}{1^2 2!}\frac{(-S_2)^0}{2^0 0!} + \frac{S_1^0}{1^0 0!}\frac{(-S_2)^1}{2^1 1!} \\ &= \frac{S_1^2}{2} + \frac{-S_2}{2} \\ &= \frac{1}{2}S_1^2 - \frac{1}{2}S_2, \end{aligned}$$

$$\begin{aligned} e_3 &= \sum_{\substack{k_1,k_2,k_3\ge 0\\ k_1+2k_2+3k_3=3}} \frac{S_1^{k_1}}{1^{k_1}k_1!}\frac{(-S_2)^{k_2}}{2^{k_2}k_2!}\frac{S_3^{k_3}}{3^{k_3}k_3!} \\ &= \frac{S_1^3}{1^3 3!}\frac{(-S_2)^0}{2^0 0!}\frac{S_3^0}{3^0 0!} + \frac{S_1^1}{1^1 1!}\frac{(-S_2)^1}{2^1 1!}\frac{S_3^0}{3^0 0!} + \frac{S_1^0}{1^0 0!}\frac{(-S_2)^0}{2^0 0!}\frac{S_3^1}{3^1 1!} \\ &= \frac{S_1^3}{3!} + S_1\frac{-S_2}{2} + \frac{S_3}{3} \\ &= \frac{1}{6}S_1^3 - \frac{1}{2}S_1S_2 + \frac{1}{3}S_3, \end{aligned}$$

and

$$\begin{aligned} e_4 &= \sum_{\substack{k_1,k_2,k_3,k_4\ge 0\\ k_1+2k_2+3k_3+4k_4=4}} \frac{S_1^{k_1}}{1^{k_1}k_1!}\frac{(-S_2)^{k_2}}{2^{k_2}k_2!}\frac{S_3^{k_3}}{3^{k_3}k_3!}\frac{(-S_4)^{k_4}}{4^{k_4}k_4!} \\ &= \frac{S_1^4}{1^4 4!}\frac{(-S_2)^0}{2^0 0!}\frac{S_3^0}{3^0 0!}\frac{(-S_4)^0}{4^0 0!} + \frac{S_1^2}{1^2 2!}\frac{(-S_2)^1}{2^1 1!}\frac{S_3^0}{3^0 0!}\frac{(-S_4)^0}{4^0 0!} \\ &\quad + \frac{S_1^1}{1^1 1!}\frac{(-S_2)^0}{2^0 0!}\frac{S_3^1}{3^1 1!}\frac{(-S_4)^0}{4^0 0!} + \frac{S_1^0}{1^0 0!}\frac{(-S_2)^2}{2^2 2!}\frac{S_3^0}{3^0 0!}\frac{(-S_4)^0}{4^0 0!} + \frac{S_1^0}{1^0 0!}\frac{(-S_2)^0}{2^0 0!}\frac{S_3^0}{3^0 0!}\frac{(-S_4)^1}{4^1 1!} \\ &= \frac{S_1^4}{4!} + \frac{S_1^2}{2}\frac{-S_2}{2} + S_1\frac{S_3}{3} + \frac{S_2^2}{4\cdot 2!} + \frac{-S_4}{4} \\ &= \frac{1}{24}S_1^4 - \frac{1}{4}S_1^2S_2 + \frac{1}{3}S_1S_3 + \frac{1}{8}S_2^2 - \frac{1}{4}S_4. \end{aligned}$$

[Isaac Newton, *Arithmetica Universalis* (1707); D. J. Struik, *Source Book in Mathematics* (Harvard University Press, 1969), 94–95]

▶ **11.** [*M25*] Equation (39) can also be used to express the S's in term of the h's: We find $S_1 = h_1$, $S_2 = 2h_2 - h_1^2$, $S_3 = 3h_3 - 3h_1h_2 + h_1^3$, etc. What is the coefficient of $h_1^{k_1}h_2^{k_2}\ldots h_m^{k_m}$ in this representation of S_m, when $k_1 + 2k_2 + \cdots mk_m = m$?

From Eq. (39),

$$h_n = \frac{1}{n}\sum_{1\le k\le n} S_k h_{n-k}$$

for $n \geq 1$; or equivalently,

$$\begin{aligned}
h_n = \frac{1}{n} \sum_{1 \leq k \leq n} S_k h_{n-k} \quad &\iff \quad nh_n = \sum_{1 \leq k \leq n} S_k h_{n-k} \\
&\iff \quad nh_n = S_n h_0 + \sum_{1 \leq k \leq n-1} S_k h_{n-k} \\
&\iff \quad S_n h_0 = nh_n - \sum_{1 \leq k \leq n-1} S_k h_{n-k} \\
&\iff \quad S_n = nh_n - \sum_{1 \leq k \leq n-1} S_k h_{n-k}.
\end{aligned}$$

Note that for $G(z) = \sum_{k \geq 0} h_k z^k$,

$$\begin{aligned}
\sum_{k \geq 1} \frac{S_k z^k}{k} &= \ln G(z) && \text{by Eq. (37)} \\
&= \ln \sum_{k \geq 0} h_k z^k \\
&= \ln \left(h_0 z^0 + \sum_{k \geq 1} h_k z^k \right) \\
&= \ln \left(1 + \sum_{k \geq 1} h_k z^k \right) \\
&= \sum_{m \geq 1} \frac{(-1)^{m+1}}{m} \left(\sum_{k \geq 1} h_k z^k \right)^m && \text{by Eq. (24)} \\
&= \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \left(\sum_{m \geq 1} h_m z^m \right)^k .
\end{aligned}$$

By the *multinomial theorem*[1], Eqs. 1.2.6-(42) and 1.2.6-(41),

$$\begin{aligned}
&\sum_{m\ge1}\frac{S_m z^m}{m}\\
&= \sum_{m\ge1}\frac{(-1)^{m+1}}{m}\left(\sum_{k\ge1}h_k z^k\right)^m\\
&= \sum_{m\ge1}\frac{(-1)^{m-1}}{m}\left(\sum_{1\le k\le m}h_k z^k\right)^m\\
&= \sum_{m\ge1}\frac{(-1)^{m-1}}{m}\left(\sum_{1\le k\le m}h_k z^k\right)^m\\
&= \sum_{m\ge1}\frac{(-1)^{m-1}}{m}\sum_{\substack{k_1,k_2,\ldots,k_m\ge0\\k_1+k_2+\cdots+k_m=m}}\binom{m}{k_1,k_2,\ldots,k_m}\prod_{1\le j\le m}\left(h_j z^j\right)^{k_j}\\
&= \sum_{m\ge1}\frac{(-1)^{m-1}}{m}\sum_{\substack{k_1,k_2,\ldots,k_m\ge0\\k_1+k_2+\cdots+k_m=m}}\frac{m!}{k_1!k_2!\ldots k_m!}\prod_{1\le j\le m}\left(h_j z^j\right)^{k_j}\\
&= \sum_{m\ge1}\sum_{\substack{k_1,k_2,\ldots,k_m\ge0\\k_1+k_2+\cdots+k_m=m}}\frac{(-1)^{m-1}(m-1)!}{k_1!k_2!\ldots k_m!}\prod_{1\le j\le m}h_j^{k_j}z^{jk_j}\\
&= \sum_{m\ge1}\sum_{\substack{k_1,k_2,\ldots,k_m\ge0\\k_1+k_2+\cdots+k_m=m}}\frac{(-1)^{k_1+k_2+\cdots+k_m-1}(k_1+k_2+\cdots+k_m-1)!}{k_1!k_2!\ldots k_m!}\\
&\quad h_1^{k_1}h_2^{k_2}\cdots h_m^{k_m}z^{k_1+2k_2+\cdots+mk_m}\\
&= \sum_{m\ge1}\sum_{\substack{k_1,k_2,\ldots,k_m\ge0\\k_1+2k_2+\cdots+mk_m=m}}\frac{(-1)^{k_1+k_2+\cdots+k_m-1}(k_1+k_2+\cdots+k_m-1)!}{k_1!k_2!\ldots k_m!}h_1^{k_1}h_2^{k_2}\cdots h_m^{k_m}z^m.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{m\ge1}S_m z^m\\
&= \sum_{m\ge1}\sum_{\substack{k_1,k_2,\ldots,k_m\ge0\\k_1+2k_2+\cdots+mk_m=m}}\frac{(-1)^{k_1+k_2+\cdots+k_m-1}m(k_1+k_2+\cdots+k_m-1)!}{k_1!k_2!\cdots k_m!}h_1^{k_1}h_2^{k_2}\cdots h_m^{k_m}z^m.
\end{aligned}$$

That is, the coefficient of $h_1^{k_1}h_2^{k_2}\cdots h_m^{k_m}$ in this representation of S_m, when $k_1+2k_2+\cdots mk_m=m$ is

$$(-1)^{k_1+k_2+\cdots+k_m-1}m(k_1+k_2+\cdots+k_m-1)!/k_1!k_2!\cdots k_m!.$$

[Albert Girard, *Invention Nouvelle en Algébre* (Amsterdam: 1629)]

▶ **12.** [*M20*] Suppose we have a doubly scripted sequence $\langle a_{mn}\rangle$ for $m,n=0,1,\ldots$; show how this double sequence can be represented by a *single* generating function of two variables, and determine the generating function for $\langle\binom{n}{m}\rangle$.

[1] See H. Richter, Ein einfacher Beweis der Newtonschen und der Waringschen Formel für die Potenzsummen, *Archiv der Mathematik* **2** (1949) 1–4.

Given a sequence $\langle a_{m,n} \rangle = \langle \binom{n}{m} \rangle$ for $m, n \geq 0$, we find the generating function to be

$$\begin{aligned}
\sum_{m,n\geq 0} a_{m,n} w^m z^n &= \sum_{m,n\geq 0} \binom{n}{m} w^m z^n \\
&= \sum_{n\geq 0} \sum_{m\geq 0} \binom{n}{m} w^m z^n \\
&= \sum_{n\geq 0} (1+w)^n z^n && \text{by Eq. (19)} \\
&= \sum_{n\geq 0} (z+wz)^n \\
&= 1/(1-(z+wz)) && \text{by Eq. (5)} \\
&= 1/(1-z-wz).
\end{aligned}$$

13. [*HM22*] The *Laplace transform* of a function $f(x)$ is the function

$$\mathbf{L}f(s) = \int_0^\infty e^{-st} f(t)dt.$$

Given that $a_0, a_1, a_2, \ldots$ is an infinite sequence having a convergent generating function, let $f(x)$ be the step function $\sum_k a_k[0 \leq k \leq x]$. Express the Laplace transform of $f(x)$ in terms of the generating function G for this sequence.

Given the step function $f(x) = \sum_{0\leq k\leq x} a_k$ and generating function $G(z)$ of $\langle a_n \rangle$ as

$$G(z) = \sum_{n\geq 0} a_n z^n,$$

we have that

$$\begin{aligned}
\mathbf{L}f(s) &= \int_0^\infty e^{-st} f(t)dt \\
&= \sum_{n\geq 0} \int_n^{n+1} e^{-st} f(t)dt \\
&= \sum_{n\geq 0} \int_n^{n+1} f(t)e^{-st}dt \\
&= \sum_{n\geq 0} \int_n^{n+1} \sum_{0\leq k\leq n} a_k e^{-st}dt \\
&= \sum_{n\geq 0} \sum_{0\leq k\leq n} \int_n^{n+1} a_k e^{-st}dt \\
&= \sum_{n\geq 0} \sum_{0\leq k\leq n} a_k \int_n^{n+1} e^{-st}dt \\
&= \sum_{n\geq 0} f(n) \int_n^{n+1} e^{-st}dt \\
&= \sum_{n\geq 0} f(n) \left. \frac{-1}{s} e^{-st} \right|_n^{n+1} \\
&= \sum_{n\geq 0} f(n)(\frac{-1}{s} e^{-s(n+1)} - \frac{-1}{s} e^{-sn}) \\
&= \sum_{n\geq 0} f(n)(e^{-sn} - e^{-s(n+1)})/s \\
&= \sum_{n\geq 0} f(n)e^{-sn}/s - \sum_{n\geq 0} f(n)e^{-s(n+1)}/s \\
&= \sum_{n\geq 0} f(n)e^{-sn}/s - \sum_{n\geq 1} f(n-1)e^{-sn}/s \\
&= \sum_{n\geq 0} f(n)e^{-sn}/s - \sum_{n\geq 0} f(n-1)e^{-sn}/s \\
&= \sum_{n\geq 0} (f(n) - f(n-1))e^{-sn}/s \\
&= \sum_{n\geq 0} a_n e^{-sn}/s \\
&= \sum_{n\geq 0} a_n (e^{-s})^n/s \\
&= G(e^{-s})/s.
\end{aligned}$$

14. [*HM21*] Prove Eq. (13).

We may prove Eq. (13).

Proposition. $\sum_{\substack{n\geq 0 \\ n \bmod m = r}} a_n z^n = \frac{1}{m} \sum_{0\leq k<m} \omega^{-kr} G(\omega^k z)$.

Proof. Let m and r be arbitrary integers such that $0 \leq r < m$, and let $\omega = e^{2\pi i/m} = \cos(2\pi/m) + i\sin(2\pi/m)$. If $G(z)$ is the generating function for $\langle a_n \rangle = a_0, a_1, \ldots$, we must show that

$$\sum_{\substack{n>0 \\ n \bmod m=r}} a_n z^n = \frac{1}{m} \sum_{0\le k<m} \omega^{-kr} G(\omega^k z).$$

But given $\omega = e^{2\pi i/m}$ and the sum of the geometric progression for n restricted such that $n \bmod m = r$,

$$\begin{aligned}
\sum_{\substack{n>0 \\ n \bmod m=r}} a_n z^n &= \sum_{\substack{n>0 \\ n-r\equiv 0 \pmod m}} a_n z^n \\
&= \sum_{n>0} [n-r \equiv 0 \pmod m] a_n z^n \\
&= \frac{m}{m} \sum_{n>0} [n-r \equiv 0 \pmod m] a_n z^n \\
&= \frac{1}{m} \sum_{n>0} a_n z^n [n-r \equiv 0 \pmod m] m \\
&= \frac{1}{m} \sum_{n>0} a_n z^n [n-r \equiv 0 \pmod m] \sum_{0\le k<m} 1^k \\
&= \frac{1}{m} \sum_{n>0} a_n z^n \sum_{0\le k<m} (1^{(n-r)/m})^k \\
&= \frac{1}{m} \sum_{n>0} a_n z^n \sum_{0\le k<m} ((-1)^{2(n-r)/m})^k \\
&= \frac{1}{m} \sum_{n>0} a_n z^n \sum_{0\le k<m} (e^{2\pi i(n-r)/m})^k \\
&= \frac{1}{m} \sum_{n>0} a_n z^n \sum_{0\le k<m} (\omega^{n-r})^k \\
&= \frac{1}{m} \sum_{n>0} a_n z^n \sum_{0\le k<m} \omega^{k(n-r)} \\
&= \frac{1}{m} \sum_{n>0} a_n z^n \sum_{0\le k<m} \omega^{k(n-r)} \\
&= \frac{1}{m} \sum_{n>0} a_n z^n \sum_{0\le k<m} \omega^{kn-kr} \\
&= \frac{1}{m} \sum_{n>0} a_n z^n \sum_{0\le k<m} \omega^{-kr} \omega^{kn} \\
&= \frac{1}{m} \sum_{0\le k<m} \omega^{-kr} \sum_{n>0} a_n \omega^{kn} z^n \\
&= \frac{1}{m} \sum_{0\le k<m} \omega^{-kr} \sum_{n>0} a_n (\omega^k z)^n \\
&= \frac{1}{m} \sum_{0\le k<m} \omega^{-kr} G(\omega^k z),
\end{aligned}$$

and hence the result. □

[exercise 1.2.6-38]

15. [*M28*] By considering $H(w) = \sum_{n \geq 0} G_n(z) w^n$, find a closed form for the generating function

$$G_n(z) = \sum_{k=0}^{n} \binom{n-k}{k} z^k = \sum_{k=0}^{n} \binom{2k-n-1}{k} (-z)^k.$$

In the case that $n > 0$, we have that

$$\begin{aligned}
G_n(z) &= \sum_{0 \leq k \leq n} \binom{n-k}{k} z^k \\
&= \sum_{0 \leq k \leq n} \left(\binom{n-k-1}{k} + \binom{n-k-1}{k-1} \right) z^k && \text{by 1.2.6-(9)} \\
&= \sum_{0 \leq k \leq n} \binom{n-k-1}{k} z^k + \sum_{0 \leq k \leq n} \binom{n-k-1}{k-1} z^k \\
&= \binom{n-n-1}{n} z^n + \binom{n-n-1}{n-1} z^n \\
&\quad + \sum_{0 \leq k \leq n-1} \binom{n-k-1}{k} z^k + \sum_{0 \leq k \leq n-1} \binom{n-k-1}{k-1} z^k \\
&= 0 + 0 \\
&\quad + \sum_{0 \leq k \leq n-1} \binom{n-k-1}{k} z^k + \sum_{0 \leq k \leq n-1} \binom{n-k-1}{k-1} z^k \\
&= \sum_{0 \leq k \leq n-1} \binom{n-k-1}{k} z^k + \sum_{0 \leq k \leq n-1} \binom{n-k-1}{k-1} z^k \\
&= \sum_{0 \leq k \leq n-1} \binom{n-k-1}{k} z^k + \sum_{1 \leq k \leq n-1} \binom{n-k-1}{k-1} z^k + \binom{n-1}{-1} z^0 \\
&= \sum_{0 \leq k \leq n-1} \binom{n-k-1}{k} z^k + \sum_{1 \leq k \leq n-1} \binom{n-k-1}{k-1} z^k + 0 \\
&= \sum_{0 \leq k \leq n-1} \binom{n-k-1}{k} z^k + \sum_{1 \leq k \leq n-1} \binom{n-k-1}{k-1} z^k \\
&= \sum_{0 \leq k \leq n-1} \binom{n-k-1}{k} z^k + \sum_{0 \leq k \leq n-2} \binom{n-k-2}{k} z^{k+1} \\
&= \sum_{0 \leq k \leq n-1} \binom{(n-1)-k}{k} z^k + z \sum_{0 \leq k \leq n-2} \binom{(n-2)-k}{k} z^k \\
&= \sum_{0 \leq k \leq n-1} \binom{(n-1)-k}{k} z^k + z \sum_{0 \leq k \leq n-2} \binom{(n-2)-k}{k} z^k + \delta n, 0; \\
&= G_{n-1}(z) + z G_{n-2}(z) + \delta_{n,0};
\end{aligned}$$

and in the case that $n = 0$, we have that

$$\begin{aligned}
G_n(z) &= \sum_{0\le k\le 0} \binom{0-k}{k} z^k \\
&= \binom{0-0}{0} z^0 \\
&= \binom{0}{0} z^0 \\
&= 1 \\
&= \delta n, 0 \\
&= \sum_{0\le k\le n-1} \binom{(n-1)-k}{k} z^k + z \sum_{0\le k\le n-2} \binom{(n-2)-k}{k} z^k + \delta n, 0 \\
&= G_{n-1}(z) + zG_{n-2}(z) + \delta_{n,0}.
\end{aligned}$$

That is, in either case,

$$G_n(z) = G_{n-1}(z) + zG_{n-2}(z) + \delta_{n,0}.$$

Then,

$$\begin{aligned}
H(w) &= \sum_{n\ge 0} G_n(z) w^n \\
&= G_0(z) + G_1(z)w + G_2(z)w^2 + \cdots, \\
wH(w) &= G_0(z)w + G_1(z)w^2 + G_2(z)w^3 + \cdots, \\
zw^2H(w) &= zG_0(z)w^2 + zG_1(z)w^3 + zG_2(z)w^4 + \cdots;
\end{aligned}$$

so that

$$\begin{aligned}
&H(w) - wH(w) - zw^2H(w) \\
&\quad = H(w)(1 - w - zw^2) \\
&\quad = G_0(z) + (G_1(z) - G_0(z))w + (G_2(z) - (G_1(z) + zG_0(z)))w^2 + \cdots \\
&\quad = G_0(z) + (G_1(z) - G_0(z))w + (G_2(z) - (G_1(z) + zG_0(z) + \delta_{2,0}))w^2 + \cdots \\
&\quad = G_0(z) + (G_1(z) - G_0(z))w + (G_2(z) - G_2(z))w^2 + \cdots \\
&\quad = G_0(z) + (G_1(z) - G_0(z))w \\
&\quad = G_0(z) + G_1(z)w - G_0(z)w \\
&\quad = G_0(z) + (G_0(z) + zG_{-1}(z) + \delta_{1,0})w - G_0(z)w \\
&\quad = G_0(z) + G_0(z)w - G_0(z)w \\
&\quad = G_0(z), \\
&\quad = 1,
\end{aligned}$$

or equivalently, so that

$$H(w) = 1/(1 - w - zw^2).$$

Then, if $z \neq -\frac{1}{4}$,

$$
\begin{aligned}
&H(w)\\
&\quad = \sum_{n\geq 0} G_n(z)w^n\\
&\quad = 1/(1-w-zw^2)\\
&\quad = \frac{1}{1-\frac{2}{2}w-\frac{\sqrt{1+4z}}{2}w+\frac{\sqrt{1+4z}}{2}w-\frac{4z}{4}w^2}\\
&\quad = \frac{1}{1-\frac{1}{2}w-\frac{\sqrt{1+4z}}{2}w-\frac{1}{2}w+\frac{\sqrt{1+4z}}{2}w+\frac{1-1-4z}{4}w^2}\\
&\quad = \frac{1}{1-\frac{1}{2}(1-\sqrt{1+4z})w-\frac{1}{2}(1+\sqrt{1+4z})w+\frac{1}{4}(1+\sqrt{1+4z}-\sqrt{1+4z}-(1+4z))w^2}\\
&\quad = \frac{1}{1-\frac{1}{2}(1-\sqrt{1+4z})w-\frac{1}{2}(1+\sqrt{1+4z})w+\frac{1}{4}(1+\sqrt{1+4z})(1-\sqrt{1+4z})w^2}\\
&\quad = \frac{4x^2}{4x^2-2x^2(1-\sqrt{1+4z})w-2x^2(1+\sqrt{1+4z})w+x^2(1+\sqrt{1+4z})(1-\sqrt{1+4z})w^2}\\
&\quad = \frac{(1+\sqrt{1+4z})(2\sqrt{1+4z}-\sqrt{1+4z}(1-\sqrt{1+4z})w)}{(2\sqrt{1+4z}-\sqrt{1+4z}(1+\sqrt{1+4z})w)(2\sqrt{1+4z}-\sqrt{1+4z}(1-\sqrt{1+4z})w)}\\
&\qquad - \frac{(1-\sqrt{1+4z})(2\sqrt{1+4z}-\sqrt{1+4z}(1+\sqrt{1+4z})w)}{(2\sqrt{1+4z}-\sqrt{1+4z}(1+\sqrt{1+4z})w)(2\sqrt{1+4z}-\sqrt{1+4z}(1-\sqrt{1+4z})w)}\\
&\quad = \frac{1+\sqrt{1+4z}}{2\sqrt{1+4z}-\sqrt{1+4z}(1+\sqrt{1+4z})w}-\frac{1-\sqrt{1+4z}}{2\sqrt{1+4z}-\sqrt{1+4z}(1-\sqrt{1+4z})w}\\
&\quad = \frac{1+\sqrt{1+4z}}{2\sqrt{1+4z}-\sqrt{1+4z}(1+\sqrt{1+4z})w}-\frac{1-\sqrt{1+4z}}{2\sqrt{1+4z}-\sqrt{1+4z}(1-\sqrt{1+4z})w}\\
&\quad = \frac{1}{\sqrt{1+4z}}\left(\frac{1+\sqrt{1+4z}}{2}\right)\frac{1}{1-\frac{1+\sqrt{1+4z}}{2}w}-\frac{1}{\sqrt{1+4z}}\left(\frac{1-\sqrt{1+4z}}{2}\right)\frac{1}{1-\frac{1-\sqrt{1+4z}}{2}w}\\
&\quad = \frac{1}{\sqrt{1+4z}}\left(\frac{1+\sqrt{1+4z}}{2}\right)\sum_{n\geq 0}\left(\frac{1+\sqrt{1+4z}}{2}\right)^n w^n\\
&\qquad - \frac{1}{\sqrt{1+4z}}\left(\frac{1-\sqrt{1+4z}}{2}\right)\sum_{n\geq 0}\left(\frac{1-\sqrt{1+4z}}{2}\right)^n w^n \qquad \text{by Eq. (5)}\\
&\quad = \frac{1}{\sqrt{1+4z}}\left(\sum_{n\geq 0}\left(\frac{1+\sqrt{1+4z}}{2}\right)^{n+1} w^n-\sum_{n\geq 0}\left(\frac{1-\sqrt{1+4z}}{2}\right)^{n+1} w^n\right)\\
&\quad = \sum_{n\geq 0}\frac{1}{\sqrt{1+4z}}\left(\left(\frac{1+\sqrt{1+4z}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{1+4z}}{2}\right)^{n+1}\right)w^n
\end{aligned}
$$

if and only if

$$G_n(z) = \left(\left(\frac{1+\sqrt{1+4z}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{1+4z}}{2}\right)^{n+1}\right)\Big/\sqrt{1+4z};$$

and if $z = -\frac{1}{4}$,

$$\begin{aligned}\sum_{n\geq 0} G_n\left(-\frac{1}{4}\right) w^n &= 1\Big/\left(1 - w + \frac{1}{4}w^2\right)\\ &= \frac{1}{\frac{1}{4}w^2 - \frac{1}{2}w - \frac{1}{2}w + 1}\\ &= \frac{1}{\left(\frac{w}{2} - 1\right)^2}\\ &= \frac{1}{\left(1 - \frac{w}{2}\right)^2}\\ &= \left(\frac{1}{1 - \frac{w}{2}}\right)^2\\ &= \left(\sum_{n\geq 0} \frac{1}{2^n} w^n\right)^2 && \text{by Eq. (5)}\\ &= \sum_{n\geq 0}\sum_{0\leq k\leq n} \frac{1}{2^k} w^k \frac{1}{2^{n-k}} w^{n-k} && \text{by Eq. (6)}\\ &= \sum_{n\geq 0}\sum_{0\leq k\leq n} \frac{1}{2^n} w^n\\ &= \sum_{n\geq 0} \frac{1}{2^n} w^n \sum_{0\leq k\leq n} 1\\ &= \sum_{n\geq 0} \frac{1}{2^n} w^n (n+1)\\ &= \sum_{n\geq 0} \frac{(n+1)}{2^n} w^n,\end{aligned}$$

so that for $n \geq 0$,

$$G_n\left(-\frac{1}{4}\right) = (n+1)/2^n.$$

Hence, for $n \geq 0$,

$$G_n(z) = \begin{cases}\left(\left(\frac{1+\sqrt{1+4z}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{1+4z}}{2}\right)^{n+1}\right)\Big/\sqrt{1+4z} & \text{if } z \neq -\frac{1}{4}\\ (n+1)/2^n & \text{otherwise.}\end{cases}$$

16. [*M22*] Give a simple formula for the generating function $G_{nr}(z) = \sum_k a_{nkr} z^k$, where a_{nkr} is the number of ways to choose k out of n objects, subject to the condition that each object may be chosen at most r times. (If $r = 1$, we have $\binom{n}{k}$ ways, and if $r \geq k$, we have the number of combinations with repetitions as in exercise 1.2.6-60.)

We want to find $a_{n,k,r}$, the number of ways to choose k out of n objects, subject to the condition that each object may be chosen at most r times.

Letting z be the formal parameter of our generating function $G_{n,r}(z)$, the number of ways to choose an object zero times is $[z^0]G_{1,0}(z) = 1$; the number of ways to choose an object one time is $[z^1]G_{1,1} = 1$; and the number of ways to choose an object r times is $[z^r]G_{1,r} = 1$. That is, $G_{1,r}(z) = 1 + z + \cdots + z^r$. For n classes of objects, this is then given by the generating function

$$G_{n,r}(z) = (1 + z + \cdots + z^r)^n = (\frac{1 - z^{r+1}}{1 - z})^n.$$

In the case that $r = 1$, we have that

$$G_{n,1}(z) = (1+z)^n = \sum_k \binom{n}{k} z^k,$$

or equivalently that $a_{n,k,1} = \binom{n}{k}$; and in the case that $r \geq k$,

$$\begin{aligned}
G_{n,r}(z) &= (1 + z + \cdots)^n \\
&= (\frac{1}{1-z})^n \\
&= (1-z)^{-n} \\
&= \sum_k \binom{-n}{k} (-z)^k \\
&= \sum_k (-1)^k \binom{k-(n+k-1)-1}{k} z^k \\
&= \sum_k \binom{n+k-1}{k} z^k && \text{by Eq. 1.2.6-(17)}
\end{aligned}$$

as shown in exercise 1.2.6-60, or equivalently that $a_{n,k,r} = \binom{n+k-1}{k}$ for $r \geq k$, $r \to \infty$.

[exercise 1.2.6-60]

17. [*M25*] What are the coefficients of $1/(1-z)^w$ if this function is expanded into a *double* power series in terms of both z and w?

We have

$$\begin{aligned}
1/(1-z)^w &= 1/(1-z)^{(w-1)+1} \\
&= \sum_k \binom{-(w-1)-1}{k} (-z)^k && \text{by Eq. (20)} \\
&= \sum_k \binom{-w}{k} (-z)^k \\
&= \sum_k (-1)^k \binom{-w}{k} z^k \\
&= \sum_k (-1)^k \left((-w)^{\underline{k}}/k!\right) z^k && \text{by Eq. 1.2.6-(3)} \\
&= \sum_k (-1)^k (-w)^{\underline{k}} z^k/k! \\
&= \sum_k w^{\overline{k}} z^k/k! && \text{by Eq. 1.2.5-(20)} \\
&= \sum_k \prod_{0 \leq n \leq k-1} (w+n) z^k/k! && \text{by Eq. 1.2.5-(19)} \\
&= \sum_k \sum_n \left[{k \atop n}\right] w^n z^k/k! && \text{by Eq. (27)} \\
&= \sum_{k,n} \left[{k \atop n}\right] w^n z^k/k!.
\end{aligned}$$

▶ **18.** [*M25*] Given positive integers n and r, find a simple formula for the value of the following sums: (a) $\sum_{1\le k_1<k_2<\cdots<k_r\le n} k_1k_2\ldots k_r$; (b) $\sum_{1\le k_1\le k_2\le\cdots\le k_r\le n} k_1k_2\ldots k_r$. (For example, when $n=3$ and $r=2$ the sums are, respectively, $1\cdot 2+1\cdot 3+2\cdot 3$ and $1\cdot 1+1\cdot 2+1\cdot 3+2\cdot 2+2\cdot 3+3\cdot 3$.)

We may find simple formulas for the sums, given positive integers n and r.

a) For $\sum_{1\le k_1<k_2<\cdots<k_r\le n} k_1k_2\ldots k_r$ we have

$$\begin{aligned}
G_n(z) &= \sum_{j\ge 0}\sum_{1\le k_1<\cdots<k_r\le n} k_1\cdots k_r z^j \\
&= \prod_{1\le j\le n}(1+jz) && \text{by exercise 10} \\
&= z^{n+1}\prod_{0\le j\le (n+1)-1}\left(\frac{1}{z}+j\right) \\
&= z^{n+1}\sum_k \left[{n+1 \atop k}\right]\left(\frac{1}{z}\right)^k && \text{by Eq. (27)} \\
&= z^{n+1}\sum_k \left[{n+1 \atop k}\right] z^{-k} \\
&= \sum_k \left[{n+1 \atop k}\right] z^{n+1-k} \\
&= \sum_r \left[{n+1 \atop n+1-r}\right] z^r,
\end{aligned}$$

so that the formula is given by

$$\sum_{1\le k_1<k_2<\cdots<k_r\le n} k_1k_2\ldots k_r = \left[{n+1 \atop n+1-r}\right].$$

b) For $\sum_{1\le k_1\le k_2\le\cdots\le k_r\le n} k_1k_2\ldots k_r$ we have

$$\begin{aligned}
G_n(z) &= \sum_{j\ge 0}\sum_{1\le k_1\le\cdots\le k_r\le n} k_1\cdots k_r z^j \\
&= \prod_{1\le j\le n}\frac{1}{1-jz} && \text{by exercise 7} \\
&= \frac{1}{\prod_{1\le j\le n}(1-jz)} \\
&= z^{-n}\frac{z^n}{\prod_{1\le j\le n}(1-jz)} \\
&= z^{-n}\sum_k \left\{{k \atop n}\right\} z^k && \text{by Eq. (28)} \\
&= \sum_k \left\{{k \atop n}\right\} z^{k-n} \\
&= \sum_r \left\{{n+r \atop n}\right\} z^r,
\end{aligned}$$

so that the formula is given by

$$\sum_{1\le k_1\le k_2\le\cdots\le k_r\le n} k_1k_2\ldots k_r = \left\{{n+r \atop n}\right\}.$$

19. [*HM32*] (C. F. Gauss, 1812.) The sums of the following infinite series are well known:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\ln 2; \qquad 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4};$$

$$1-\frac{1}{4}+\frac{1}{7}-\frac{1}{10}+\cdots=\frac{\pi\sqrt{3}}{9}+\frac{1}{3}\ln 2.$$

Using the definition

$$H_x=\sum_{n\ge 1}\left(\frac{1}{n}-\frac{1}{n+x}\right)$$

found in the answer to exercise 1.2.7-24, these series may be written respectively as

$$1-\frac{1}{2}H_{1/2}; \qquad \frac{2}{3}-\frac{1}{4}H_{1/4}+\frac{1}{4}H_{3/4}; \qquad \frac{3}{4}-\frac{1}{6}H_{1/6}+\frac{1}{6}H_{2/3}.$$

Prove that, in general, $H_{p/q}$ has the value

$$\frac{q}{p}-\frac{\pi}{2}\cot\frac{p}{q}\pi-\ln 2q+2\sum_{0<k<q/2}\cos\frac{2pk}{q}\pi\cdot\ln\sin\frac{k}{q}\pi,$$

where p and q are integers with $0<p<q$. [*Hint:* By Abel's limit theorem the sum is

$$\lim_{x\to 1-}\sum_{n\ge 1}\left(\frac{1}{n}-\frac{1}{n+p/q}\right)x^{p+nq}.$$

Use Eq. (13) to express this power series in such a way that the limit can be evaluated.]

Proposition. *$H_{p/q}$ has the value*

$$H_{p/q}=\frac{q}{p}-\frac{\pi}{2}\cot\frac{p}{q}\pi-\ln 2q+2\sum_{0<k<q/2}\cos\frac{2pk}{q}\pi\cdot\ln\sin\frac{k}{q}\pi,$$

when p and q are integers with $0<p<q$.

Proof. As a preliminary identity, observe that

$$\begin{aligned}
\ln(1-e^{i\theta}) &= \ln\left(2e^{i(\theta-\pi)/2}\frac{e^{i\theta/2}-e^{-i\theta/2}}{2i}\right)\\
&= \ln 2+\ln\left(e^{i(\theta-\pi)/2}\right)+\ln\frac{e^{i\theta/2}-e^{-i\theta/2}}{2i}\\
&= \ln 2+\frac{1}{2}i(\theta-\pi)+\ln\frac{e^{i\theta/2}-e^{-i\theta/2}}{2i}\\
&= \ln 2+\frac{1}{2}i(\theta-\pi)+\ln\sin\frac{\theta}{2}. \qquad (19.1)
\end{aligned}$$

Let p,q be artibtrary integers such that $0<p<q$, and define H_x as

$$H_x=\sum_{n\ge 1}\left(\frac{1}{n}-\frac{1}{n+x}\right)$$

for any nonnegative rational number x. We must show that

$$H_{p/q}=\frac{q}{p}-\frac{\pi}{2}\cot\frac{p}{q}\pi-\ln 2q+2\sum_{0<k<q/2}\cos\frac{2pk}{q}\pi\cdot\ln\sin\frac{k}{q}\pi.$$

We have

$$H_{p/q} = \sum_{n\geq 1} \left(\frac{1}{n} - \frac{1}{n+p/q} \right).$$

By Abel's limit theorem,

$$\sum_{n\geq 1} \left(\frac{1}{n} - \frac{1}{n+p/q} \right) = \lim_{x\to 1-} \sum_{n\geq 1} \left(\frac{1}{n} - \frac{1}{n+p/q} \right) x^{p+nq}.$$

Then,

$$\begin{aligned}
&\lim_{x\to 1-} \sum_{n\geq 1} \left(\frac{1}{n} - \frac{1}{n+p/q} \right) x^{p+nq} \\
&\quad = \lim_{x\to 1-} \left(\sum_{n\geq 1} \frac{1}{n} x^{p+nq} - \sum_{n\geq 1} \frac{1}{n+p/q} x^{p+nq} \right).
\end{aligned}$$

For the left sum,

$$\begin{aligned}
&\sum_{n\geq 1} \frac{1}{n} x^{p+nq} \\
&\quad = x^p \sum_{n\geq 1} \frac{1}{n} (x^q)^n \\
&\quad = -x^p \sum_{n\geq 1} \frac{(-1)^{2n+1}}{n} (x^q)^n \\
&\quad = -x^p \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} (-x^q)^n \\
&\quad = -x^p \ln(1-x^q). && \text{by Eq. (24)}
\end{aligned}$$

For the right sum with $\omega = e^{2\pi i/q}$,

$$\begin{aligned}
&-\sum_{n\geq 1} \frac{1}{n+p/q} x^{p+nq} \\
&= -\sum_{\substack{n>0 \\ p+nq \bmod q=p}} \frac{q}{p+nq} x^{p+nq} \\
&= \frac{q}{p}x^p - \sum_{\substack{n\geq 0 \\ p+nq \bmod q=p}} \frac{q}{p+nq} x^{p+nq} \\
&= \frac{q}{p}x^p - \frac{1}{q}\sum_{0\leq k<q} \omega^{-kp} \sum_{n\geq 1} \frac{q}{n}\omega^{kn}x^n && \text{by Eq. (13)} \\
&= \frac{q}{p}x^p - \sum_{0\leq k<q} \omega^{-kp} \sum_{n\geq 1} \frac{1}{n}(\omega^k x)^n \\
&= \frac{q}{p}x^p + \sum_{0\leq k<q} \omega^{-kp} \sum_{n\geq 1} \frac{(-1)^{2n+1}}{n}(\omega^k x)^n \\
&= \frac{q}{p}x^p + \sum_{0\leq k<q} \omega^{-kp} \sum_{n\geq 1} \frac{(-1)^{n+1}}{n}(-\omega^k x)^n \\
&= \frac{q}{p}x^p + \sum_{0\leq k<q} \omega^{-kp} \ln(1-\omega^k x). && \text{by Eq. (24)}
\end{aligned}$$

That is,

$$H_{p/q} = \lim_{x\to 1-} \left(\sum_{0\leq k<q} \omega^{-kp} \ln(1-\omega^k x) - x^p \ln(1-x^q) + \frac{q}{p}x^p \right).$$

Continuing in the limit as $x \to 1-$,

$$\begin{aligned}
&\sum_{0\leq k<q} \omega^{-kp} \ln(1-\omega^k x) - x^p \ln(1-x^q) + \frac{q}{p}x^p \\
&= \sum_{1\leq k<q} \omega^{-kp} \ln(1-\omega^k x) + \ln(1-x) - x^p \ln(1-x^q) + \frac{q}{p}x^p \\
&= \sum_{1\leq k<q} \omega^{-kp} \ln(1-\omega^k x) + \ln(1-x) + \frac{q}{p}x^p - x^p \ln(1-x^q) + x^p \ln(1-x) - x^p \ln(1-x) \\
&= \sum_{1\leq k<q} \omega^{-kp} \ln(1-\omega^k x) + (1-x^p)\ln(1-x) + \frac{q}{p}x^p - x^p \ln\frac{1-x^q}{1-x}. \quad (19.2)
\end{aligned}$$

The limit of the first term is

$$\begin{aligned}
&\lim_{x\to 1-} \sum_{1\le k<q} \omega^{-kp} \ln(1-\omega^k x) \\
&\quad = \sum_{1\le k<q} \omega^{-kp} \ln(1-\omega^k) \\
&\quad = \sum_{1\le k<q} \omega^{-kp} \left(\ln 2 + \frac{1}{2} i(2\pi k/q - \pi) + \ln\sin\frac{2\pi k/q}{2}\right) \qquad \text{by (19.1)} \\
&\quad = \sum_{1\le k<q} \omega^{-kp} \left(\ln 2 + \frac{i\pi k}{q} - \frac{i\pi}{2} + \ln\sin\frac{k}{q}\pi\right) \\
&\quad = \sum_{1\le k<q} \omega^{-kp} \left(\ln 2 + \frac{i\pi k}{q} - \frac{i\pi}{2}\right) + \sum_{1\le k<q} \omega^{-kp} \ln\sin\frac{k}{q}\pi,
\end{aligned}$$

and the limit of the remaining terms of (19.2) is

$$\lim_{x\to 1-} (1-x^p)\ln(1-x) + \frac{q}{p}x^p - x^p \ln\frac{1-x^q}{1-x} = \frac{q}{p} - \ln q;$$

but

$$
\begin{aligned}
&\sum_{1\le k<q} \omega^{-kp}\left(\ln 2 + \frac{i\pi k}{q} - \frac{i\pi}{2}\right) \\
&\quad = \sum_{1\le k<q} \omega^{-kp}\ln 2 + \sum_{1\le k<q} \omega^{-kp}\frac{i\pi k}{q} - \sum_{1\le k<q} \omega^{-kp}\frac{i\pi}{2} \\
&\quad = \frac{\ln 2(\omega^{-p(q-1)}-1)}{\omega^p-1} + \sum_{1\le k<q} \omega^{-kp}\frac{i\pi k}{q} - \sum_{1\le k<q} \omega^{-kp}\frac{i\pi}{2} \\
&\quad = \frac{\ln 2(\omega^{-p(q-1)}-1)}{\omega^p-1} + \frac{i\pi\omega^{-p(q-1)}(q(-\omega^p)+\omega^{pq}+q-1)}{q(\omega^p-1)^2} - \sum_{1\le k<q} \omega^{-kp}\frac{i\pi}{2} \\
&\quad = \frac{\ln 2(\omega^{-p(q-1)}-1)}{\omega^p-1} + \frac{i\pi\omega^{-p(q-1)}(q(-\omega^p)+\omega^{pq}+q-1)}{q(\omega^p-1)^2} - \frac{i\pi(\omega^{-p(q-1)}-1)}{2(\omega^p-1)} \\
&\quad = \frac{-\ln 2(\omega^p-1)}{\omega^p-1} + \frac{i\pi\omega^{-p(q-1)}(q(-\omega^p)+\omega^{pq}+q-1)}{q(\omega^p-1)^2} - \frac{i\pi(\omega^{-p(q-1)}-1)}{2(\omega^p-1)} \\
&\quad = -\ln 2 + \frac{i\pi\omega^{-p(q-1)}(q(-\omega^p)+\omega^{pq}+q-1)}{q(\omega^p-1)^2} - \frac{i\pi(\omega^{-p(q-1)}-1)}{2(\omega^p-1)} \\
&\quad = -\ln 2 + \frac{i\pi\omega^{-p(q-1)}(q(-\omega^p)+\omega^{pq}+q-1)}{q(\omega^p-1)^2} + \frac{i\pi(\omega^p-1)}{2(\omega^p-1)} \\
&\quad = -\ln 2 + \frac{i\pi\omega^{-p(q-1)}(q(-\omega^p)+\omega^{pq}+q-1)}{q(\omega^p-1)^2} + \frac{i\pi}{2} \\
&\quad = -\ln 2 + \frac{i\pi}{2} + \frac{i\pi\omega^{-p(q-1)}(q(-\omega^p)+\omega^{pq}+q-1)}{q(\omega^p-1)^2} \\
&\quad = -\ln 2 + \frac{i\pi}{2} + \frac{i\pi\omega^{-p(q-1)}(q(-\omega^p)+q)}{q(\omega^p-1)^2} \\
&\quad = -\ln 2 + \frac{i\pi}{2} + \frac{i\pi\omega^{-p(q-1)}(-\omega^p+1)}{(\omega^p-1)^2} \\
&\quad = -\ln 2 + \frac{i\pi}{2} - \frac{i\pi\omega^p(\omega^p-1)}{(\omega^p-1)^2} \\
&\quad = -\ln 2 + \frac{i\pi}{2} - \frac{i\pi\omega^p}{\omega^p-1} \\
&\quad = -\ln 2 + \frac{i\pi}{2} - \frac{i\pi}{-\omega^{-p}+1} \\
&\quad = -\ln 2 + \frac{i\pi}{2} + \frac{i\pi}{\omega^{-p}-1},
\end{aligned}
$$

and

$$\begin{aligned}
&\sum_{1\le k<q} \omega^{-kp} \ln\sin\frac{k}{q}\pi \\
&\quad= \sum_{1\le k<q/2} \left(\omega^{-kp} + \omega^{-(q-k)p}\right) \ln\sin\frac{k}{q}\pi \\
&\quad= \sum_{1\le k<q/2} 2\omega^{-pq/2} \cos\left(\frac{\pi p(q-2k)}{q}\right) \ln\sin\frac{k}{q}\pi \\
&\quad= 2\sum_{1\le k<q/2} \cos\left(\frac{\pi p(q-2k)}{q}\right) \ln\sin\frac{k}{q}\pi \\
&\quad= 2\sum_{1\le k<q/2} \cos\left(\frac{2\pi pq}{2q} - \frac{2\pi pk}{q}\right) \ln\sin\frac{k}{q}\pi \\
&\quad= 2\sum_{1\le k<q/2} \cos\left(-\frac{2\pi pk}{q}\right) \ln\sin\frac{k}{q}\pi \\
&\quad= 2\sum_{0<k<q/2} \cos\frac{2pk}{q}\pi \cdot \ln\sin\frac{k}{p}\pi.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\frac{i}{2} + \frac{i}{\omega^{-p}-1} \\
&\quad= -\frac{1}{2}\cot\frac{p}{q}\pi.
\end{aligned}$$

Hence,

$$H_{p/q} = \frac{q}{p} - \frac{\pi}{2}\cot\frac{p}{q}\pi - \ln 2q + 2\sum_{0<k<q/2} \cos\frac{2pk}{q}\pi \cdot \ln\sin\frac{k}{q}\pi,$$

as we needed to show.

□

[exercise 1.2.7-24; C. F. Gauss, §33 of his monograph on hypergeometric series, Eq. [75]; Abel, *Crelle* **1** (1826), 314–315]

20. [*M21*] For what coefficients c_{mk} is $\sum_{n\ge 0} n^m z^n = \sum_{k=0}^{m} c_{mk} z^k/(1-z)^{k+1}$?

First, observe that multiplying both sides of Eq. (20) by z^n yields

$$\begin{aligned}\frac{z^n}{(1-z)^{n+1}} &= z^n \sum_{k\geq 0} \binom{n+k}{n} z^k \\ &= \sum_{k\geq 0} \binom{n+k}{n} z^{n+k} \\ &= \sum_{n\geq 0} \binom{n}{k} z^n.\end{aligned}$$

We then have

$$\begin{aligned}\sum_{n\geq 0} n^m z^n &= \sum_{n\geq 0} \sum_k {m \brace k} n^{\underline{k}} z^n && \text{by Eq. 1.2.6-(45)} \\ &= \sum_k {m \brace k} \sum_{n\geq 0} n^{\underline{k}} z^n \\ &= \sum_k k! {m \brace k} \sum_{n\geq 0} \frac{n^{\underline{k}}}{k!} z^n \\ &= \sum_k k! {m \brace k} \sum_{n\geq 0} \binom{n}{k} z^n \\ &= \sum_k k! {m \brace k} \frac{z^n}{(1-z)^{n+1}},\end{aligned}$$

so that

$$c_{m,k} = k! {m \brace k}.$$

21. [*HM30*] Set up the generating function for the sequence $\langle n! \rangle$ and study properties of this function.

Let $G(z) = \sum_{n\geq 0} n! z^n$ be the generating function for the sequence $\langle n! \rangle$. Given the recurrence

$$n! = n(n-1)! + [n = 0]$$

we have that

$$\begin{aligned}
G(z) &= \sum_{n\geq 0} n!z^n \\
&= \sum_{n\geq 0} n(n-1)!z^n + \sum_{n=0} z^n \\
&= \sum_{n\geq 0} n(n-1)!z^n + 1 \\
&= \sum_{n\geq 0} (n+1)n!z^{n+1} + 1 \\
&= \sum_{n\geq 0} (n)n!z^{n+1} + \sum_{n\geq 0} n!z^{n+1} + 1 \\
&= \sum_{n\geq 0} (n)n!z^{n+1} + z\sum_{n\geq 0} n!z^n + 1 \\
&= \sum_{n\geq 0} (n)n!z^{n+1} + zG(z) + 1 \\
&= \sum_{n\geq 0} (n+1)(n+1)!z^{n+2} + zG(z) + 1 \\
&= z^2\sum_{n\geq 0} (n+1)(n+1)!z^n + zG(z) + 1 \\
&= z^2G'(z) + zG(z) + 1. && \text{by Eq. (14)}
\end{aligned}$$

This is the ordinary differential equation

$$z^2G'(z) + (z-1)G(z) + 1 = 0,$$

satisfied by

$$G(z) = \frac{-e^{-1/z}}{z}\left(E_1(-1/z) + C\right),$$

for the *exponential integral* $E_1(z) = \int_z^\infty \frac{1}{te^t}dt$ and constant C; that is,

$$G(z) = \frac{-e^{-1/z}}{z}\left(\int_{-1/z}^\infty \frac{1}{te^t}dt + C\right),$$

since

$$
\begin{aligned}
\frac{d}{dz}G(z) &= \frac{d}{dz}\left(\frac{-e^{-1/z}}{z}\left(E_1(-1/z)+C\right)\right)\\
&= \frac{-e^{-1/z}}{z}\frac{d}{dz}\left(E_1(-1/z)+C\right)+\frac{d}{dz}\frac{-e^{-1/z}}{z}\left(E_1(-1/z)+C\right)\\
&= \frac{-e^{-1/z}}{z}\frac{d}{dz}\left(E_1(-1/z)+C\right)+\frac{\frac{d}{dz}\left(-e^{-1/z}\right)z+e^{-1/z}\frac{d}{dz}z}{z^2}\left(E_1(-1/z)+C\right)\\
&= \frac{-e^{-1/z}}{z}\frac{d}{dz}\left(E_1(-1/z)+C\right)+\frac{-e^{-1/z}\frac{d}{dz}\left(-1/z\right)z+e^{-1/z}}{z^2}\left(E_1(-1/z)+C\right)\\
&= \frac{-e^{-1/z}}{z}\frac{d}{dz}\left(E_1(-1/z)+C\right)+\frac{-e^{-1/z}\left(1/z^2\right)z+e^{-1/z}}{z^2}\left(E_1(-1/z)+C\right)\\
&= \frac{-e^{-1/z}}{z}\frac{d}{dz}\left(E_1(-1/z)+C\right)+\frac{-e^{-1/z}\left(1/z\right)+e^{-1/z}}{z^2}\left(E_1(-1/z)+C\right)\\
&= \frac{-e^{-1/z}}{z}\frac{d}{dz}\left(E_1(-1/z)+C\right)+\frac{-e^{-1/z}+ze^{-1/z}}{z^3}\left(E_1(-1/z)+C\right)\\
&= \frac{-e^{-1/z}}{z}\frac{d}{dz}\left(E_1(-1/z)+C\right)+\frac{(z-1)e^{-1/z}}{z^3}\left(E_1(-1/z)+C\right)\\
&= \frac{-e^{-1/z}}{z}\frac{d}{dz}\left(E_1(-1/z)+C\right)-\frac{z-1}{z^2}\frac{-e^{-1/z}}{z}\left(E_1(-1/z)+C\right)\\
&= \frac{-e^{-1/z}}{z}\frac{d}{dz}\left(E_1(-1/z)+C\right)-\frac{z-1}{z^2}G(z)\\
&= \frac{-e^{-1/z}}{z}\frac{d}{dz}\left(\int_{-1/z}^{\infty}\frac{1}{te^t}dt+C\right)-\frac{z-1}{z^2}G(z)\\
&= \frac{-e^{-1/z}}{z}ze^{1/z}\frac{d}{dz}\frac{-1}{z}-\frac{z-1}{z^2}G(z)\\
&= \frac{-e^{-1/z}}{z}ze^{1/z}\frac{1}{z^2}-\frac{z-1}{z^2}G(z)\\
&= \frac{-e^{-1/z}}{z}e^{1/z}\frac{1}{z}-\frac{z-1}{z^2}G(z)\\
&= \frac{-e^{-1/z}}{z}\frac{e^{1/z}}{z}-\frac{z-1}{z^2}G(z)\\
&= -\frac{1}{z^2}-\frac{z-1}{z^2}G(z)\\
&= -\frac{1}{z^2}\left(1+(z-1)G(z)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&z^2G'(z)+(z-1)G(z)+1\\
&\quad= z^2\left(-\frac{1}{z^2}\left(1+(z-1)G(z)\right)\right)+(z-1)G(z)+1\\
&\quad= -\left(1+(z-1)G(z)\right)+(z-1)G(z)+1\\
&\quad= -1-(z-1)G(z)+(z-1)G(z)+1\\
&\quad= 0.
\end{aligned}
$$

This generating function diverges, as may be verified using the root test, since for positive n,

$$\begin{aligned}
\sqrt[n]{n!} &\le \sqrt[n]{\frac{n^{n+1}}{e^{n-1}}} && \text{by exercise 1.2.5-24} \\
&\approx \sqrt[n]{\frac{n^n}{e^n}} \\
&= \frac{n}{e}
\end{aligned}$$

is unbounded.

[K. Knopp, *Infinite Sequences and Series* (Dover, 1956), Section 66 [sic]]

22. [*M21*] Find a generating function $G(z)$ for which

$$[z^n]G(z) = \sum_{k_0+2k_1+4k_2+8k_3+\cdots=n} \binom{r}{k_0}\binom{r}{k_1}\binom{r}{k_2}\binom{r}{k_3}\cdots.$$

As a preliminary, observe that if z is an arbitrary complex number such that $|z| < 1$, we have that

$$\prod_{i\ge 0}\left(1+z^{2^i}\right) = \frac{1}{1-z},$$

since

$$\begin{aligned}
(1-z)\prod_{i\ge 0}\left(1+z^{2^i}\right) &= (1-z)\lim_{j\to\infty}\prod_{i=0}^{j}\left(1+z^{2^i}\right) \\
&= \lim_{j\to\infty}(1-z)\prod_{i=0}^{j}\left(1+z^{2^i}\right) \\
&= \lim_{j\to\infty}(1-z)(1+z)\prod_{i=1}^{j}\left(1+z^{2^i}\right) \\
&= \lim_{j\to\infty}\left(1-z^2\right)\prod_{i=1}^{j}\left(1+z^{2^i}\right) \\
&= \lim_{j\to\infty}\left(1-z^2\right)\left(1+z^2\right)\prod_{i=2}^{j}\left(1+z^{2^i}\right) \\
&= \lim_{j\to\infty}\left(1-z^{2^2}\right)\prod_{i=2}^{j}\left(1+z^{2^i}\right) \\
&= \cdots \\
&= \lim_{j\to\infty}\left(1-z^{2^j}\right)\left(1+z^{2^j}\right) \\
&= \lim_{j\to\infty} 1-z^{2^{j+1}} \\
&= 1.
\end{aligned}$$

Now, for the given sum,

$$\sum_{2^0k_0+2^1k_1+2^2k_2+\cdots=n} \binom{r}{k_0}\binom{r}{k_1}\binom{r}{k_2}\cdots,$$

we may generate the $\binom{r}{k_i}$ terms for all $i \geq 0$ using the binomial theorem as

$$\begin{aligned}\sum_{k_i\geq 0}\binom{r}{k_i}z^{2^i k_i} &= \sum_{k_i\geq 0}\binom{r}{k_i}(z^{2^i})^{k_i}\\ &= \left(1+z^{2^i}\right)^r,\end{aligned}$$

and then multiply them together to arrive at the equivalence

$$\begin{aligned}&\sum_{2^0k_0+2^1k_1+2^2k_2+\cdots=n} \binom{r}{k_0}\binom{r}{k_1}\binom{r}{k_2}\cdots\\ &= \left(1+z^{2^0}\right)^r\left(1+z^{2^1}\right)^r\left(1+z^{2^2}\right)^r\cdots\\ &= \prod_{i\geq 0}\left(1+z^{2^i}\right)^r\\ &= \left(\prod_{i\geq 0}\left(1+z^{2^i}\right)\right)^r\\ &= \left(\frac{1}{1-z}\right)^r\\ &= \frac{1}{(1-z)^r}\\ &= (1-z)^{-r}.\end{aligned}$$

That is, the generating function is

$$G(z) = (1-z)^{-r}.$$

Incidentally,

$$\begin{aligned}(1-z)^{-r} &= (-z+1)^{-r}\\ &= \sum_n \binom{-r}{n}(-z)^n 1^{-r-n} && \text{by Eq. 1.2.6-(13)}\\ &= \sum_n (-1)^n\binom{-r}{n}z^n\\ &= \sum_n \binom{n-(-r)-1}{n}z^n && \text{by Eq. 1.2.6-(17)}\\ &= \sum_n \binom{r+n-1}{n}z^n.\end{aligned}$$

23. [*M33*] (L. Carlitz.) (a) Prove that for all integers $m \geq 1$ there are polynomials $f_m(z_1, \ldots, z_m)$ and $g_m(z_1, \ldots, z_m)$ such that the formula

$$\sum_{k_1,\ldots,k_m \geq 0} \binom{r}{n-k_1}\binom{k_1}{n-k_2}\cdots\binom{k_{m-1}}{n-k_m} z_1^{k_1}\ldots z_m^{k_m} = f_m(z_1,\ldots,z_m)^{n-r} g_m(z_1,\ldots,z_m)^r$$

is an identity for all integers $n \geq r \geq 0$.

(b) Generalizing exercise 15, find a closed form for the sum

$$S_n(z_1,\ldots,z_m) = \sum_{k_1,\ldots,k_m \geq 0} \binom{k_1}{n-k_2}\binom{k_2}{n-k_3}\cdots\binom{k_m}{n-k_1} z_1^{k_1}\ldots z_m^{k_m}$$

in terms of the functions f_m and g_m in part (a).

(c) Find a simple expression for $S_n(z_1, \ldots, z_m)$ when $z_1 = \cdots = z_m = z$.

The answers to exercise 23 follow below.

(a) We may prove the identity.

Proposition. *For all integers $m \geq 1$ there are polynomials $f_m(z_1, \ldots, z_m)$ and $g_m(z_1, \ldots, z_m)$ such that for all integers $n \geq r \geq 0$,*

$$\sum_{k_1,\ldots,k_m \geq 0} \binom{r}{n-k_1}\binom{k_1}{n-k_2}\cdots\binom{k_{m-1}}{n-k_m} z_1^{k_1}\ldots z_m^{k_m}$$
$$= f_m(z_1,\ldots,z_m)^{n-r} g_m(z_1,\ldots,z_m)^r.$$

Proof. Let m be an arbitrary positive integer. We must show that there are polynomials $f_m(z_1, \ldots, z_m)$ and $g_m(z_1, \ldots, z_m)$ such that for all integers $n \geq r \geq 0$,

$$\sum_{k_1,\ldots,k_m \geq 0} \binom{r}{n-k_1}\binom{k_1}{n-k_2}\cdots\binom{k_{m-1}}{n-k_m} z_1^{k_1}\ldots z_m^{k_m}$$
$$= f_m(z_1,\ldots,z_m)^{n-r} g_m(z_1,\ldots,z_m)^r.$$

We proceed with a proof by induction on m.

In the case that $m = 1$, let $f_1(z_1) = z_1$ and $g_1(z_1) = z_1 + 1$. Then we must show in this case that

$$\sum_{k_1 \geq 0} \binom{r}{n-k_1} z_1^{k_1} = f_1(z_1)^{n-r} g_1(z_1)^r$$
$$= z_1^{n-r}(z_1+1)^r.$$

If $r = 0$,

$$\begin{aligned}\sum_{k_1 \geq 0} \binom{0}{n-k_1} z_1^{k_1} &= \sum_{k_1 = n} \binom{0}{n-k_1} z_1^{k_1} \\ &= \binom{0}{0} z_1^n \\ &= z_1^n \\ &= z_1^{n-0}(z_1+1)^0.\end{aligned}$$

Then, assuming

$$\sum_{k_1 \geq 0} \binom{r}{n-k_1} z_1^{k_1} = z_1^{n-r}(z_1+1)^r,$$

we must show that

$$\sum_{k_1 \geq 0} \binom{r+1}{n-k_1} z_1^{k_1} = z_1^{n-(r+1)}(z_1+1)^{r+1}.$$

But

$$\begin{aligned}\sum_{k_1 \geq 0} \binom{r+1}{n-k_1} z_1^{k_1} &= \sum_{k_1 \geq 0} \binom{r}{n-k_1} z_1^{k_1} + \sum_{k_1 \geq 0} \binom{r}{n-k_1-1} z_1^{k_1} \\ &= \sum_{k_1 \geq 0} \binom{r}{n-k_1} z_1^{k_1} + \sum_{k_1 \geq 0} \binom{r}{(n-1)-k_1} z_1^{k_1} \\ &= z_1^{n-r}(z_1+1)^r + z_1^{(n-1)-r}(z_1+1)^r \\ &= \left(z_1^{n-r} + z_1^{(n-1)-r}\right)(z_1+1)^r \\ &= \left(z_1^{n-r} + z_1^{n-(r+1)}\right)(z_1+1)^r \\ &= \left(z_1 z_1^{n-(r+1)} + z_1^{n-(r+1)}\right)(z_1+1)^r \\ &= z_1^{n-(r+1)}(z_1+1)(z_1+1)^r \\ &= z_1^{n-(r+1)}(z_1+1)^{r+1}.\end{aligned}$$

As our inductive hypothesis, we assume that there are polynomials $f_m(z_1, \ldots, z_m)$ and $g_m(z_1, \ldots, z_m)$ such that for all integers $n \geq r \geq 0$,

$$\begin{aligned}&\sum_{k_1,\ldots,k_m \geq 0} \binom{r}{n-k_1}\binom{k_1}{n-k_2}\cdots\binom{k_{m-1}}{n-k_m} z_1^{k_1}\ldots z_m^{k_m} \\ &\quad = f_m(z_1,\ldots,z_m)^{n-r} g_m(z_1,\ldots,z_m)^r.\end{aligned}$$

We must show that there are polynomials $f_{m+1}(z_1, \ldots, z_{m+1})$ and $g_{m+1}(z_1, \ldots, z_{m+1})$ such that for all integers $n \ge r \ge 0$,

$$\sum_{k_1,\ldots,k_{m+1}\ge 0} \binom{r}{n-k_1}\binom{k_1}{n-k_2}\cdots\binom{k_{(m+1)-1}}{n-k_{m+1}} z_1^{k_1}\ldots z_{m+1}^{k_{m+1}} \\ = f_{m+1}(z_1,\ldots,z_{m+1})^{n-r} g_{m+1}(z_1,\ldots,z_{m+1})^r.$$

But, set $z_m \leftarrow z_m(1+z_{m+1}^{-1})$, and let

$$f_{m+1}(z_1,\ldots,z_{m+1}) = z_{m+1} f_m(z_1,\ldots,z_m(1+z_{m+1}^{-1})),$$
$$g_{m+1}(z_1,\ldots,z_{m+1}) = z_{m+1} g_m(z_1,\ldots,z_m(1+z_{m+1}^{-1})).$$

Then

$$\begin{aligned}
&f_{m+1}(z_1,\ldots,z_{m+1})^{n-r} g_{m+1}(z_1,\ldots,z_{m+1})^r \\
&\quad= \left(z_{m+1} f_m(z_1,\ldots,z_m(1+z_{m+1}^{-1}))\right)^{n-r} \left(z_{m+1} g_m(z_1,\ldots,z_m(1+z_{m+1}^{-1}))\right)^r \\
&\quad= z_{m+1}^n f_m(z_1,\ldots,z_m(1+z_{m+1}^{-1}))^{n-r} g_m(z_1,\ldots,z_m(1+z_{m+1}^{-1}))^r \\
&\quad= z_{m+1}^n \sum_{k_1,\ldots,k_m\ge 0} \binom{r}{n-k_1}\binom{k_1}{n-k_2}\cdots\binom{k_{m-1}}{n-k_m} z_1^{k_1}\ldots \left(z_m(1+z_{m+1}^{-1})\right)^{k_m} \\
&\quad= z_{m+1}^n \sum_{k_1,\ldots,k_m\ge 0} \binom{r}{n-k_1}\binom{k_1}{n-k_2}\cdots\binom{k_{m-1}}{n-k_m} z_1^{k_1}\ldots z_m^{k_m}(1+z_{m+1}^{-1})^{k_m} \\
&\quad= z_{m+1}^n \sum_{k_1,\ldots,k_m\ge 0} \binom{r}{n-k_1}\binom{k_1}{n-k_2}\cdots\binom{k_{m-1}}{n-k_m} z_1^{k_1}\ldots z_m^{k_m} \\
&\qquad\cdot \sum_{k_{m+1}\ge 0} \binom{k_m}{k_{m+1}} 1^{k_m-k_{m+1}} z_{m+1}^{-k_{m+1}} \\
&\quad= \sum_{k_1,\ldots,k_m\ge 0} \binom{r}{n-k_1}\binom{k_1}{n-k_2}\cdots\binom{k_{m-1}}{n-k_m} z_1^{k_1}\ldots z_m^{k_m} \\
&\qquad\cdot \sum_{k_{m+1}\ge 0} \binom{k_m}{k_{m+1}} z_{m+1}^{n-k_{m+1}} \\
&\quad= \sum_{k_1,\ldots,k_m\ge 0} \binom{r}{n-k_1}\binom{k_1}{n-k_2}\cdots\binom{k_{m-1}}{n-k_m} z_1^{k_1}\ldots z_m^{k_m} \\
&\qquad\cdot \sum_{k_{m+1}\ge 0} \binom{k_{(m+1)-1}}{n-k_{m+1}} z_{m+1}^{k_{m+1}} \\
&\quad= \sum_{k_1,\ldots,k_{m+1}\ge 0} \binom{r}{n-k_1}\binom{k_1}{n-k_2}\cdots\binom{k_{(m+1)-1}}{n-k_{m+1}} z_1^{k_1}\ldots z_{m+1}^{k_{m+1}}.
\end{aligned}$$

This is what we needed to show. □

Note that $f_m(z_1, \ldots, z_m)$ obeys the recurrence

$$f_m(z_1,\ldots,z_m) = \begin{cases} 0 & \text{if } m < 0 \\ 1 & \text{if } m = 0 \\ z_m f_{m-1} + z_{m-1} f_{m-2} & \text{otherwise,} \end{cases}$$

since $f_1(z_1) = z_1 = z_1 f_0 + z_0 f_{-1}$, and if $f_m(z_1, \ldots, z_m) = z_m f_{m-1} + z_{m-1} f_{m-2}$, then

$$\begin{aligned} f_{m+1}(z_1, \ldots, z_{m+1}) &= z_{m+1} f_m(z_1, \ldots, z_{m-1}, z_m(1 + z_{m+1}^{-1})) \\ &= z_{m+1}(z_m(1 + z_{m+1}^{-1}) f_{m-1} + z_{m-1} f_{m-2}) \\ &= z_{m+1}(z_m z_{m+1}^{-1} f_{m-1} + z_m f_{m-1} + z_{m-1} f_{m-2}) \\ &= z_{m+1}(z_m z_{m+1}^{-1} f_{m-1} + f_m) \\ &= z_m z_{m+1} z_{m+1}^{-1} f_{m-1} + z_{m+1} f_m \\ &= z_{m+1} f_m + z_m f_{m-1}; \end{aligned}$$

and similarly $g_m(z_1, \ldots, z_m)$ the recurrence

$$g_m(z_1, \ldots, z_m) = \begin{cases} 1 & \text{if } m \le 0 \\ z_m g_{m-1} + z_{m-1} g_{m-2} & \text{otherwise,} \end{cases}$$

since $g_1(z_1) = z_1 + 1 = z_1 g_0 + z_0 g_{-1}$, and if $g_m(z_1, \ldots, z_m) = z_m g_{m-1} + z_{m-1} g_{m-2}$, then

$$\begin{aligned} g_{m+1}(z_1, \ldots, z_{m+1}) &= z_{m+1} g_m(z_1, \ldots, z_{m-1}, z_m(1 + z_{m+1}^{-1})) \\ &= z_{m+1}(z_m(1 + z_{m+1}^{-1}) g_{m-1} + z_{m-1} g_{m-2}) \\ &= z_{m+1}(z_m z_{m+1}^{-1} g_{m-1} + z_m g_{m-1} + z_{m-1} g_{m-2}) \\ &= z_{m+1}(z_m z_{m+1}^{-1} g_{m-1} + g_m) \\ &= z_m z_{m+1} z_{m+1}^{-1} g_{m-1} + z_{m+1} g_m \\ &= z_{m+1} g_m + z_m g_{m-1}. \end{aligned}$$

(b) We want to find a closed form for the sum

$$S_n(z_1, \ldots, z_m) = \sum_{k_1, \ldots, k_m \ge 0} \binom{k_1}{n - k_2} \binom{k_2}{n - k_3} \cdots \binom{k_m}{n - k_1} z_1^{k_1} \cdots z_m^{k_m}$$

in terms of the functions f_m and g_m of part (a). But

$$\begin{aligned}
&S_n(z_1, \ldots, z^{m-1}, z) \\
&\quad = \sum_{k_1,\ldots,k_m \ge 0} \binom{k_1}{n-k_2}\binom{k_2}{n-k_3}\cdots\binom{k_m}{n-k_1} z_1^{k_1}\cdots z^{k_m} \\
&\quad = \sum_{k_1,\ldots,k_m \ge 0} \binom{k_m}{n-k_1}\binom{k_1}{n-k_2}\cdots\binom{k_{m-1}}{n-k_m} z_1^{k_1}\cdots z^{k_m} \\
&\quad = [z_m^n] \sum_{k_1,\ldots,k_m \ge 0} \binom{k_m}{n-k_1}\binom{k_1}{n-k_2}\cdots\binom{k_{m-1}}{n-k_m} z_1^{k_1}\cdots z^{k_m} z_m^n \\
&\quad = [z_m^n] \sum_{\substack{k_1,\ldots,k_m \ge 0 \\ r=k_m}} \binom{r}{n-k_1}\binom{k_1}{n-k_2}\cdots\binom{k_{m-1}}{n-k_m} z_1^{k_1}\cdots z^r z_m^{n-r+k_m} \\
&\quad = [z_m^n] \sum_{\substack{k_1,\ldots,k_m \ge 0 \\ 0\le r\le n}} \binom{r}{n-k_1}\binom{k_1}{n-k_2}\cdots\binom{k_{m-1}}{n-k_m} z_1^{k_1}\cdots z^r z_m^{n-r+k_m} \\
&\quad = [z_m^n] \sum_{\substack{k_1,\ldots,k_m \ge 0 \\ 0\le r\le n}} \binom{r}{n-k_1}\binom{k_1}{n-k_2}\cdots\binom{k_{m-1}}{n-k_m} z_1^{k_1}\cdots z_m^{k_m} z^r z_m^{n-r} \\
&\quad = [z_m^n] \sum_{r=0}^{n} z^r z_m^{n-r} \sum_{k_1,\ldots,k_m \ge 0} \binom{r}{n-k_1}\binom{k_1}{n-k_2}\cdots\binom{k_{m-1}}{n-k_m} z_1^{k_1}\cdots z_m^{k_m} \\
&\quad = [z_m^n] \sum_{r=0}^{n} z^r z_m^{n-r} f_m(z_1,\ldots,z_m)^{n-r} g_m(z_1,\ldots,z_m)^r.
\end{aligned}$$

Then,

$$\begin{aligned}
&S_n(z_1, \ldots, z_m) \\
&\quad = [z_m^n] \sum_{r=0}^{n} z_m^r z_m^{n-r} f_m(z_1,\ldots,z_m)^{n-r} g_m(z_1,\ldots,z_m)^r \\
&\quad = [z_m^n] \sum_{r=0}^{n} f_m(z_1,\ldots,z_m)^{n-r} g_m(z_1,\ldots,z_m)^r z_m^n \\
&\quad = \sum_{r=0}^{n} f_m(z_1,\ldots,z_m)^{n-r} g_m(z_1,\ldots,z_m)^r \\
&\quad = \sum_{r=0}^{n} (z_m f_{m-1} + z_{m-1} f_{m-2})^{n-r} (z_m g_{m-1} + z_{m-1} g_{m-2})^r \\
&\quad = \sum_{r=0}^{n} \left(\sum_{0\le s\le n-r} \binom{n-r}{s} (z_m f_{m-1})^s (z_{m-1} f_{m-2})^{n-r-s} \sum_{0\le s\le n} \binom{r}{s} (z_{m-1} g_{m-2})^s (z_m g_{m-1})^{r-s} \right) \\
&\quad = \sum_{0\le s\le r\le n} \binom{r}{s}\binom{n-r}{s} (z_m g_{m-1})^{r-s} (z_{m-1} g_{m-2})^s (z_m f_{m-1})^s (z_{m-1} f_{m-2})^{n-r-s};
\end{aligned}$$

and, by Eq. (20),

$$
\begin{aligned}
&S_n(z_1,\ldots,z_m)\\
&\quad= \sum_{0\le s\le r\le n} \binom{r}{s}\binom{n-r}{s} (z_m g_{m-1})^{r-s}(z_{m-1}g_{m-2})^s(z_m f_{m-1})^s(z_{m-1}f_{m-2})^{n-r-s}\\
&\quad= [z^n]\sum_{s\ge 0}\sum_{r\ge s}\sum_{n\ge r} \binom{r}{s}\binom{n-r}{s} (z_m g_{m-1}z)^{r-s}(z_{m-1}g_{m-2}z)^s(z_m f_{m-1}z)^s(z_{m-1}f_{m-2}z)^{n-r-s}\\
&\quad= [z^n]\sum_{s\ge 0}(z_{m-1}g_{m-2}z)^s(z_m f_{m-1}z)^s\sum_{r\ge s}\binom{r}{s}(z_m g_{m-1}z)^{r-s}\sum_{n\ge r}\binom{n-r}{s}(z_{m-1}f_{m-2}z)^{n-r-s}\\
&\quad= [z^n]\sum_{s\ge 0}(z_{m-1}g_{m-2}z)^s(z_m f_{m-1}z)^s\sum_{r\ge s}\binom{r}{s}(z_m g_{m-1}z)^{r-s}\sum_{n-r\ge 0}\binom{n-r}{s}(z_{m-1}f_{m-2}z)^{n-r-s}\\
&\quad= [z^n]\sum_{s\ge 0}(z_{m-1}g_{m-2}z)^s(z_m f_{m-1}z)^s\sum_{r\ge s}\binom{r}{s}(z_m g_{m-1}z)^{r-s}\sum_{n\ge 0}\binom{n}{s}(z_{m-1}f_{m-2}z)^{n-s}\\
&\quad= [z^n]\sum_{s\ge 0}(z_{m-1}g_{m-2}z)^s(z_m f_{m-1}z)^s\sum_{r\ge s}\binom{r}{s}(z_m g_{m-1}z)^{r-s}(z_{m-1}f_{m-2}z)^{-s}\sum_{n\ge 0}\binom{n}{s}(z_{m-1}f_{m-2}z)^{n}\\
&\quad= [z^n]\sum_{s\ge 0}(z_{m-1}g_{m-2}z)^s(z_m f_{m-1}z)^s\sum_{r\ge s}\binom{r}{s}(z_m g_{m-1}z)^{r-s}(z_{m-1}f_{m-2}z)^{-s}\frac{(z_{m-1}f_{m-2}z)^s}{(1-z_{m-1}f_{m-2}z)^{s+1}}\\
&\quad= [z^n]\sum_{s\ge 0}\frac{(z_{m-1}g_{m-2}z)^s(z_m f_{m-1}z)^s}{(1-z_{m-1}f_{m-2}z)^{s+1}}\sum_{r\ge s}\binom{r}{s}(z_m g_{m-1}z)^{r-s}\\
&\quad= [z^n]\sum_{s\ge 0}\frac{(z_{m-1}g_{m-2}z)^s(z_m f_{m-1}z)^s}{(1-z_{m-1}f_{m-2}z)^{s+1}}\sum_{r-s\ge 0}\binom{s+r-s}{s}(z_m g_{m-1}z)^{r-s}\\
&\quad= [z^n]\sum_{s\ge 0}\frac{(z_{m-1}g_{m-2}z)^s(z_m f_{m-1}z)^s}{(1-z_{m-1}f_{m-2}z)^{s+1}}\sum_{r\ge 0}\binom{s+r}{s}(z_m g_{m-1}z)^{r}\\
&\quad= [z^n]\sum_{s\ge 0}\frac{(z_{m-1}g_{m-2}z)^s(z_m f_{m-1}z)^s}{(1-z_{m-1}f_{m-2}z)^{s+1}}\frac{1}{(1-z_m g_{m-1}z)^{s+1}}\\
&\quad= [z^n]\sum_{s\ge 0}\frac{(z_{m-1}g_{m-2}z)^s(z_m f_{m-1}z)^s}{(1-z_m g_{m-1}z)^{s+1}(1-z_{m-1}f_{m-2}z)^{s+1}}\\
&\quad= [z^n]\sum_{s\ge 0}\frac{1}{1-z_m g_{m-1}z}\frac{1}{1-z_{m-1}f_{m-2}z}\left(\frac{z_{m-1}g_{m-2}z_m f_{m-1}z^2}{(1-z_m g_{m-1}z)(1-z_{m-1}f_{m-2}z)}\right)^s\\
&\quad= [z^n]\frac{1}{1-z_m g_{m-1}z}\frac{1}{1-z_{m-1}f_{m-2}z}\frac{1}{1-z_{m-1}g_{m-2}z_m f_{m-1}z^2/((1-z_m g_{m-1}z)(1-z_{m-1}f_{m-2}z))}\\
&\quad= [z^n]\frac{1}{z_m g_{m-1}z_{m-1}f_{m-2}z^2-z_m g_{m-1}z-z_{m-1}g_{m-2}z_m f_{m-1}z^2-z_{m-1}f_{m-2}z+1}\\
&\quad= [z^n]\frac{1}{(1-z_m g_{m-1}z)(1-z_{m-1}f_{m-2}z)-z_{m-1}g_{m-2}z_m f_{m-1}z^2}.
\end{aligned}
$$

Let

$$h_m = z_m g_{m-1} + z_{m-1} f_{m-2}$$

and

$$\begin{aligned}
&(1-\rho z)(1-\sigma z) = 1 - h_m z + (-1)^m z_1 \cdots z_m z^2 \\
&\iff \quad 1 - \rho z - \sigma z + \rho\sigma z^2 = 1 - h_m z + (-1)^m z_1 \cdots z_m z^2 \\
&\iff \quad 1 - (\rho+\sigma) z + \rho\sigma z^2 = 1 - h_m z + (-1)^m z_1 \cdots z_m z^2 \\
&\iff \quad \rho + \sigma = h_m \wedge \rho\sigma = (-1)^m z_1 \cdots z_m.
\end{aligned}$$

Then, since

$$\begin{aligned}
&z_m g_{m-1} z_{m-1} f_{m-2} - z_{m-1} g_{m-2} z_m f_{m-1} \\
&\quad = z_{m-1} z_m (g_{m-1} f_{m-2} - g_{m-2} f_{m-1}) \\
&\quad = z_{m-1} z_m ((z_{m-1} g_{m-2} + z_{m-2} g_{m-3}) f_{m-2} - g_{m-2}(z_{m-1} f_{m-2} + z_{m-2} f_{m-3})) \\
&\quad = z_{m-1} z_m (z_{m-1} g_{m-2} f_{m-2} + z_{m-2} g_{m-3} f_{m-2} - g_{m-2} z_{m-1} f_{m-2} - g_{m-2} z_{m-2} f_{m-3}) \\
&\quad = z_{m-1} z_m (z_{m-2} g_{m-3} f_{m-2} - g_{m-2} z_{m-2} f_{m-3}) \\
&\quad = (-1) z_{m-2} z_{m-1} z_m (g_{m-2} f_{m-3} - g_{m-3} f_{m-2}) \\
&\quad\ \ \vdots \\
&\quad = (-1)^m z_1 \cdots z_m
\end{aligned}$$

we find

$$\begin{aligned}
&S_n(z_1,\ldots,z_m)\\
&\quad= [z^n]\frac{1}{(1-z_m g_{m-1}z)(1-z_{m-1}f_{m-2}z)-z_{m-1}g_{m-2}z_m f_{m-1}z^2}\\
&\quad= [z^n]\frac{1}{1-z_m g_{m-1}z-z_{m-1}f_{m-2}z+z_m g_{m-1}z_{m-1}f_{m-2}z^2-z_{m-1}g_{m-2}z_m f_{m-1}z^2}\\
&\quad= [z^n]\frac{1}{1-(z_m g_{m-1}+z_{m-1}f_{m-2})z+(z_m g_{m-1}z_{m-1}f_{m-2}-z_{m-1}g_{m-2}z_m f_{m-1})z^2}\\
&\quad= [z^n]\frac{1}{1-h_m z+(-1)^m z_1\cdots z_m z^2}\\
&\quad= [z^n]\frac{1}{(1-\rho z)(1-\sigma z)}\\
&\quad= [z^n]\left(\frac{\rho}{\rho-\sigma}\frac{1}{1-\rho z}+\frac{\sigma}{\sigma-\rho}\frac{1}{1-\sigma z}\right)\\
&\quad= [z^n]\left(\frac{\rho}{\rho-\sigma}\sum_{n\ge 0}\rho^n z^n+\frac{\sigma}{\sigma-\rho}\sum_{n\ge 0}\sigma^n z^n\right)\\
&\quad= [z^n]\sum_{n\ge 0}\left(\frac{\rho}{\rho-\sigma}\rho^n+\frac{\sigma}{\sigma-\rho}\sigma^n\right)z^n\\
&\quad= [z^n]\sum_{n\ge 0}\left(\frac{\rho^{n+1}}{\rho-\sigma}-\frac{\sigma^{n+1}}{\rho-\sigma}\right)z^n\\
&\quad= [z^n]\sum_{n\ge 0}\frac{\rho^{n+1}-\sigma^{n+1}}{\rho-\sigma}z^n\\
&\quad= \frac{\rho^{n+1}-\sigma^{n+1}}{\rho-\sigma}.
\end{aligned}$$

(c) When $z_1=\cdots=z_m=z$, we may simplify $S_n(z,\ldots,z)$. In the case $m=1$,

$$\begin{aligned}
\rho_1+\sigma_1 &= h_1 = z_1 = z,\\
\rho_1\sigma_1 &= (-1)^1 z_1 = -z_1 = -z,
\end{aligned}$$

satisfied by

$$\begin{aligned}
\rho_1 &= \frac{z+\sqrt{z^2+4z}}{2},\\
\sigma_1 &= \frac{z-\sqrt{z^2+4z}}{2};
\end{aligned}$$

and in the case $m \geq 1$, since

$$\begin{aligned}\rho_m + \sigma_m &= h_m \\ &= zg_{m-1} + zf_{m-2} \\ &= z(g_{m-1} + f_{m-2}) \\ &= z(zg_{m-2} + zg_{m-3} + zf_{m-3} + zf_{m-4}) \\ &= z^2(g_{m-2} + g_{m-3} + f_{m-3} + f_{m-4}) \\ &= \vdots \\ &= z^m \\ &= (\rho_1 + \sigma_1)^m\end{aligned}$$

and

$$\begin{aligned}\rho_m\sigma_m &= (-1)^m z^m \\ &= (-z)^m \\ &= (\rho_1\sigma_1)^m,\end{aligned}$$

we find in general that

$$\begin{aligned}\rho_m &= \rho_1^m, \\ \sigma_m &= \sigma_1^m.\end{aligned}$$

That is,

$$\begin{aligned}S_n(z,\ldots,z) &= \frac{\rho_m^{n+1} - \sigma_m^{n+1}}{\rho_m - \sigma_m} \\ &= \frac{\rho_1^{m(n+1)} - \sigma_1^{m(n+1)}}{\rho_1^m - \sigma_1^m} \\ &= \frac{\big((z+\sqrt{z^2+4z})/2\big)^{m(n+1)} - \big((z-\sqrt{z^2+4z})/2\big)^{m(n+1)}}{\big((z+\sqrt{z^2+4z})/2\big)^{m} - \big((z-\sqrt{z^2+4z})/2\big)^{m}} \\ &= \frac{1}{2^{mn}} \frac{(z+\sqrt{z^2+4z})^{m(n+1)} - (z-\sqrt{z^2+4z})^{m(n+1)}}{(z+\sqrt{z^2+4z})^{m} - (z-\sqrt{z^2+4z})^{m}}.\end{aligned}$$

[exercise 1.2.8-30; L. Carlitz, *Collectanea Math.* **27** [sic] (1965), 281–296]

24. [*M22*] Prove that, if $G(z)$ is any generating function, we have

$$\sum_k \binom{m}{k} [z^{n-k}]G(z)^k = [z^n](1 + zG(z))^m.$$

Evaluate both sides of this identity when $G(z)$ is (a) $1/(1-z)$; (b) $(e^x - 1)/z$.

Proposition. $\sum_k \binom{m}{k}[z^{n-k}]G(z)^k = [z^n](1+zG(z))^m$.

Proof. Suppose $G(z)$ is an arbitrary generating function,

$$G(z) = \sum_{n\geq 0} g_n z^n.$$

We must show that

$$\sum_k \binom{m}{k}[z^{n-k}]G(z)^k = [z^n](1+zG(z))^m.$$

But by a rule of the *coefficient of* operator[2],

$$\begin{aligned}\sum_k \binom{m}{k}[z^{n-k}]G(z)^k &= \sum_k \binom{m}{k}[z^n]z^k G(z)^k \\ &= [z^n]\sum_k \binom{m}{k} G(z)^k z^k \\ &= [z^n]\sum_k \binom{m}{k}\left(zG(z)\right)^k \\ &= [z^n](1+zG(z))^m,\end{aligned}$$

as we needed to show. □

(a) For $G(z) = \frac{1}{1-z}$,

$$\begin{aligned}\sum_k \binom{m}{k}[z^{n-k}]G(z)^k &= \sum_k \binom{m}{k}[z^{n-k}]\left(\frac{1}{1-z}\right)^k \\ &= \sum_k \binom{m}{k}[z^{n-k}]\frac{1}{(1-z)^k} \\ &= \sum_k \binom{m}{k}[z^{n-k}]\sum_{n\geq 0}\binom{k+n-1}{n}z^n \\ &= \sum_k \binom{m}{k}[z^{n-k}]\sum_{n-k\geq 0}\binom{k+n-k-1}{n-k}z^{n-k} \\ &= \sum_k \binom{m}{k}[z^{n-k}]\sum_{n-k\geq 0}\binom{n-1}{n-k}z^{n-k} \\ &= \sum_k \binom{m}{k}\binom{n-1}{n-k}\end{aligned}$$

[2] D. E. Knuth, *A Classical Mind* (Prentice-Hall, 1994), 248.

and

$$\begin{aligned}
[z^n](1+zG(z))^m &= [z^n]\left(1+z\frac{1}{1-z}\right)^m \\
&= [z^n]\sum_k \binom{m}{k}\left(z\frac{1}{1-z}\right)^k \\
&= [z^n]\sum_k \binom{m}{k} z^k \frac{1}{(1-z)^k} \\
&= [z^n]\sum_k \binom{m}{k} z^k \sum_n \binom{k+n-1}{n} z^n \\
&= [z^n]\sum_k \binom{m}{k} z^k \sum_{n-k} \binom{n-1}{n-k} z^{n-k} \\
&= [z^n]\sum_k \binom{m}{k} z^k \sum_k \binom{n-1}{k} z^k \\
&= [z^n](1+z)^m(1+z)^{n-1} \\
&= [z^n](1+z)^{m+n-1} \\
&= [z^n]\sum_{n\geq 0} \binom{m+n-1}{n} z^n \\
&= \binom{m+n-1}{n}.
\end{aligned}$$

(b) For $G(z) = \frac{e^z-1}{z}$,

$$\begin{aligned}
\sum_k \binom{m}{k}[z^{n-k}]G(z)^k &= \sum_k \binom{m}{k}[z^{n-k}]\left(\frac{e^z-1}{z}\right)^k \\
&= \sum_k \binom{m}{k}[z^n]z^k\left(\frac{e^z-1}{z}\right)^k \\
&= \sum_k \binom{m}{k}[z^n]z^k\frac{(e^z-1)^k}{z^k} \\
&= \sum_k \binom{m}{k}[z^n](e^z-1)^k \\
&= \sum_k \binom{m}{k}[z^n]k!\sum_n \left\{ n \atop k \right\}\frac{z^n}{n!} && \text{by Eq. (23)} \\
&= \sum_k \binom{m}{k}k![z^n]\sum_n \left\{ n \atop k \right\}\frac{z^n}{n!} \\
&= \sum_k \binom{m}{k}k!\left\{ n \atop k \right\}/n! \\
&= \sum_k \frac{m^{\underline{k}}}{k!}k!\left\{ n \atop k \right\}/n! \\
&= \sum_k m^{\underline{k}}\left\{ n \atop k \right\}/n!
\end{aligned}$$

and

$$\begin{aligned}[z^n](1+zG(z))^m &= [z^n]\left(1+z\frac{e^z-1}{z}\right)^m \\ &= [z^n]\left(e^z\right)^m \\ &= [z^n]e^{mz} \\ &= [z^n]\sum_{n\geq 0}\frac{(mz)^n}{n!} \\ &= [z^n]\sum_{n\geq 0}\frac{m^n}{n!}z^n \\ &= m^n/n!,\end{aligned}$$

since

$$\sum_k m^{\underline{k}}\left\{{n \atop k}\right\} = m^n.$$

▶ **25.** [*M23*] Evaluate the sum $\sum_k \binom{n}{k}\binom{2n-2k}{n-k}(-2)^k$ by simpliyfing the equivalent formula $\sum_k [w^k](1-2w)^n[z^{n-k}](1+z)^{2n-2k}$.

We have

$$
\begin{aligned}
&\sum_k [w^k](1-2w)^n [z^{n-k}](1+z)^{2n-2k} \\
&\quad = \sum_k [w^k](1-2w)^n [z^n] z^k (1+z)^{2n-2k} \\
&\quad = [z^n] \sum_k [w^k](1-2w)^n z^k (1+z)^{2n-2k} \\
&\quad = [z^n] \sum_k [w^k](1-2w)^n z^k (1+z)^{2n} (1+z)^{-2k} \\
&\quad = [z^n](1+z)^{2n} \sum_k [w^k](1-2w)^n z^k (1+z)^{-2k} \\
&\quad = [z^n](1+z)^{2n} \sum_k [w^k](1-2w)^n z^k (1/(1+z)^2)^k \\
&\quad = [z^n](1+z)^{2n} \sum_k [w^k](1-2w)^n (z/(1+z)^2)^k \\
&\quad = [z^n](1+z)^{2n} \sum_k [w^k](z/(1+z)^2)^k (1-2w)^n \\
&\quad = [z^n](1+z)^{2n} \sum_k [w^k](z/(1+z)^2)^k \sum_j \binom{n}{j} (-2w)^j \\
&\quad = [z^n](1+z)^{2n} \sum_k [w^k](z/(1+z)^2)^k \sum_j (-2)^j \binom{n}{j} w^j \\
&\quad = [z^n](1+z)^{2n} \sum_k (z/(1+z)^2)^k (-2)^k \binom{n}{k} \\
&\quad = [z^n](1+z)^{2n} \sum_k \binom{n}{k} (-2z/(1+z)^2)^k \\
&\quad = [z^n](1+z)^{2n} (1-2z/(1+z)^2)^n \\
&\quad = [z^n] \big((1+z)^2(1-2z/(1+z)^2)\big)^n \\
&\quad = [z^n] \big((1+z)^2 - 2z\big)^n \\
&\quad = [z^n](1+z^2)^n \\
&\quad = [z^n] \sum_k \binom{n}{k} (z^2)^k \\
&\quad = [z^n] \sum_k \binom{n}{k} z^{2k} \\
&\quad = [z^n] \sum_{\substack{k/2 \\ k \text{ even}}} \binom{n}{k/2} z^k \\
&\quad = \binom{n}{n/2} [n \text{ even}].
\end{aligned}
$$

[G. P. Egorychev, *Integral Representation and the Computation of Combintatorial Sums* (Amer. Math. Soc., 1984)]

26. [*M40*] Explore a generalization of the notation (31) according to which we might write, for example, $[z^2 - 2z^5]G(z) = a_2 - 2a_5$ when $G(z)$ is given by (1).

n.a.

[D. E. Knuth, *A Classical Mind* (Prentice-Hall, 1994), 247–258]